Estimating the discrete Lusternik–Schnirelmann category

Brian Green, Nicholas A. Scoville, and Mimi Tsuruga
November 18, 2013

Abstract
Let $K$ be a simplicial complex and suppose that $K$ collapses onto $L$. Define $n$ to be 1 minus the minimum number of collapsible sets it takes to cover $L$. Then the discrete Lusternik–Schnirelmann category of $K$ is the smallest $n$ taken over all such $L$. In this paper, we give an algorithm which yields an upper bound for the discrete category. We show our algorithm is correct and give several bounds for the discrete category of well-known simplicial complexes. We show that the discrete category of the dunce cap is 2, implying that the dunce cap is “further” from being collapsible than Bing’s house.

MSC Classification Primary 55U10, 68Q25; Secondary 57Q15, 55M30, 05E45

Keywords Lusternik–Schnirelmann category, computational topology, collapse, simplicial complex

1 Introduction
The goal of this paper is to introduce a computational algorithm to give bounds on the discrete Lusternik–Schnirelmann (LS) category of a simplicial complex. The discrete LS category was introduced in [1] as a discrete analogue of the classical LS category for topological spaces [7]. The main properties of the discrete LS category are summarized in Section 2. Our notion of the discrete LS category is based on collapsibility, which has been used to study simple homotopy type [6]. Let $f: K \to \mathbb{R}$ be a discrete Morse function in the sense of R. Forman [8, 9]. It was shown in [1] that the discrete category of $K$ bounds from below the number of critical points of $f$. There is much interest in the relationship between discrete Morse theory and collapsibility. For example, R. Ayala et al. [3] have used the collapse number of a 2-dimensional complex to study certain classes of discrete Morse functions. In addition, B. Benedetti and F. H. Lutz recently introduced so-called random discrete Morse theory [4]. They propose obtaining a discrete Morse vector by collapsing a complex until it contains no more free faces, removing a top dimensional face, and repeating.
This measurement of the complexity of a simplicial complex is in the spirit of the current work. After introducing the discrete LS category and reviewing its basic properties, we propose an algorithm to determine an upper bound for any finite type simplicial complex. We show that the algorithm is correct and discuss some experiments for several well-known simplicial complexes. Recall that a topological space is contractible if it has the homotopy type of a point. A simplicial complex which is collapsible always has a contractible geometric realization (see Proposition 2) but the converse is not true. It is well known that Bing’s house with two rooms [5] and the dunce cap [16] provide examples of complexes with contractible geometric realization, but which are not collapsible. In Proposition 6, we show that the discrete LS category of the dunce cap is 2, while the discrete category of Bing’s house is only 1, and hence the dunce cap is in a certain sense further from being collapsible than Bing’s house. This raises the question as to the existence of contractible simplicial complexes with arbitrarily large discrete category. We plan to include a downloadable version of our program\(^1\) in the near future. Suggestions for improvements are welcomed.

2 Simplicial Complexes and discrete LS category

We begin by reviewing the basic terms used throughout this paper. All simplicial complexes are assumed to be connected. Let \([n] = \{1, 2, 3, \ldots, n\}\). An abstract (finite type) simplicial complex \(K\) on \([n]\) is a collection of subsets of \([n]\) such that

1. If \(\sigma \in K\) and \(\tau \subseteq \sigma\), then \(\tau \in K\).
2. \(\{i\} \in K\) for every \(i \in [n]\).

An element \(\sigma \in K\) of cardinality \(i + 1\) is called an \(i\)-dimensional face or an \(i\)-face of \(K\). The dimension of \(K\), denoted \(\dim(K)\), is the maximum dimension over all its faces. If \(\sigma, \tau \in K\) with \(\tau \subseteq \sigma\), then \(\tau\) is a face of \(\sigma\) and \(\sigma\) is a coface of \(\tau\). We also say that \(\tau\) is a proper face of \(\sigma\) if \(\dim(\sigma) = \dim(\tau) + 1\). If \(\tau \subseteq \sigma\) and \(n = \dim(\sigma) - \dim(\tau)\), we say that \(\tau\) is of codimension \(n\) with respect to \(\sigma\). A face of \(K\) that is not contained in any other face is called a facet of \(K\). A (closed) subcomplex \(L\) of \(K\), denoted \(L \subseteq K\), is a subset \(L\) of \(K\) such that \(L\) is also a simplicial complex. Denote by \(\overline{\sigma}\) the smallest simplicial complex containing \(\sigma\). We are careful to use the term simplex for an element \(\sigma\) of \(K\) and the term complex for a subcomplex \(L\) of \(K\).

We define the boundary of \(\sigma\), denoted \(\text{bd}(\sigma)\), as the collection of all its faces. Define \(\text{cbd}(\sigma) = \{\tau \in \text{bd}(\sigma) : \tau\) has codimension 1 with respect to \(\sigma\}\). Clearly if \(\dim(\sigma) = n\), then \(|\text{cbd}(\sigma)| = n + 1\).

If \(K\) contains a pair of simplices \(\sigma, \tau\) such that \(\tau\) is a proper face of \(\sigma\) and \(\tau\) has no other cofaces, then \(K - \{\sigma, \tau\}\) is a simplicial complex called an elementary (simplicial) collapse of \(K\). The simplicial complex \(K\) is said

\(^1\)http://webpages.ursinus.edu/nscoville/research-papers.html
to collapse onto $L$ if $L$ can be obtained from $K$ through a finite series of elementary collapses, denoted $K \searrow L$. If $K$ collapses onto $L$, we also say that $L$ expands to $K$, denoted $L \nearrow K$. The pair \{\sigma, \tau\} is said to be a free pair, a term we will use to denote a pair that can either be collapsed or expanded with respect to $K$. In the case where $L = \{v\}$ is a single point, we say that $K$ is collapsible. The following is immediate from the definition of a free pair.

**Lemma 1** Let $\sigma$ be a simplex of dimension $n$, with $\tau \in \text{cbd}(\sigma)$, and $K$ a simplicial complex with $K \cap \{\sigma, \tau\} = \emptyset$. Then \{\sigma, \tau\} is a free pair of $K$ if and only if \text{cbd}(\sigma) − \tau \subseteq K.

If $K$ is a simplicial complex, denote by $|K|$ its geometric realization. Since an elementary collapse corresponds to a deformation retraction, we have the following Proposition.

**Proposition 2** [12, Proposition 6.14] If $K$ and $L$ have the same simple homotopy type, then $|K|$ and $|L|$ have the same homotopy type. In particular, if $K$ is collapsible, then $|K|$ is contractible.

**Definition 3** Let $L \subseteq K$ be a subcomplex. We say that $L$ has discrete precategory less than or equal to $m$ in $K$, denoted \(\text{dcat}_K(L) \leq m\), if there exists $m + 1$ closed subcomplexes $\{U_0, U_1, \ldots, U_m\}$, $U_i \subseteq K$ for $0 \leq i \leq m$, each of which is collapsible such that $L \subseteq \bigcup_{i=0}^{m} U_i$. If $\text{dcat}_K(L) \neq m$, then $\text{dcat}_K(L) = m$. The discrete category of $L$ in $K$ is defined by $\text{dcat}_K(L) = \min\{\text{dcat}_K(L') : L \text{ collapses onto } L'\}$. We write $\text{dcat}(K) = \text{dcat}_K(K)$. It follows immediately from the definition that if $L \searrow L'$ then $\text{dcat}_K(L) \geq \text{dcat}_K(L')$ and $\text{dcat}_K(L) \leq \text{dcat}_K(K)$.

If $G$ is a 1-dimensional complex or graph, then the discrete LS category coincides with the arboricity [10] of $G$. Nash-Williams has computed this invariant for all graphs [13]. In particular, if $G = K_n$, the complete graph on $n$ nodes, then $\text{dcat}(K_n) = \left\lceil \frac{n}{2} \right\rceil - 1$. We will make note of this fact in Table 1 when we compare the actual discrete category of $K_n$ with the estimate obtained by our algorithm.

### 3 Combinatorial lower bound

Let $K$ be a simplicial complex of dimension $n$ or $n + 1$, and $H_i(K)$ the $i^{th}$ (unreduced) simplicial homology group of $K$. Let $e_i^K$ denote the number of simplices of $K$ of dimension $i$. Recall that the **Euler characteristic** of $K$ is defined by $\chi(K) = \sum_i (-1)^i e_i^K$. If $\beta_i^K$ denotes the $i^{th}$ Betti number of $K$, it is easy to show that $\chi(K) = \sum_i (-1)^i \beta_i^K$ [14, Theorem 1.31]. Let $E(c^K) := e_0^K + e_2^K + \ldots + e_n^K$ and $O(c^K) := e_1^K + e_3^K + \ldots + e_{n+1}^K$. 


Proposition 4 Let $K$ be a simplicial complex of dimension $n$ with $c_i$ the number of simplicies of $K$ of dimension $i$, $0 \leq i \leq n$. If $E(c^K) - 1 \geq O(c^K)$, then $\left[ \frac{E(c^K) - 1}{O(c^K)} \right] - 1 \leq \text{dcat}(K)$. If $E(c^K) - 1 \leq O(c^K)$, then $\left[ \frac{O(c^K)}{E(c^K) - 1} \right] - 1 \leq \text{dcat}(K)$.

Proof We only show the first inequality, as the other one is similar. Let $U$ be a collapsible subcomplex of $K$. Since $H_i(U) = 0$ for all $i \geq 1$ whenever $U$ is collapsible [11, Corollary 2.3.5], it follows that any collapsible set must satisfy

$$\mathcal{X}(U) = 1 = \sum_i (-1)^i \beta_i^U = \sum_i (-1)^i c_i^U.$$  

Rearranging this equation yields $O(c_U) = E(c_U) - 1$. Now since $E(c^K) - 1 \geq O(c^K)$, any collapsible set $U$ of $K$ can satisfy at best $E(c_U) - 1 = O(c^K)$. Thus in order to satisfy this equation, we need at least $\left[ \frac{E(c^K) - 1}{O(c^K)} \right]$ collapsible sets.

Thus $\left[ \frac{E(c^K) - 1}{O(c^K)} \right] - 1 \leq \text{dcat}(K)$.

If $K \searrow K'$ is any elementary collapse, then $\left[ \frac{E(c^{K'}) - 1}{O(c^{K'})} \right] \leq \left[ \frac{E(c^K) - 1}{O(c^K)} \right] \leq \text{dcat}(K)$. Thus $\left[ \frac{E(c^{K'}) - 1}{O(c^{K'})} \right] \leq \text{dcat}(K)$.

\[\square\]

4 Algorithm

Let $K$ be a simplicial complex. Let $\mathcal{H}$ be a graph encoding the incidence relations of the simplicies of $K$; every node of $\mathcal{H}$ is a simplex of $K$ and there is an edge between two simplicies $\sigma, \tau$ whenever $\tau$ is a proper face of $\sigma$. This graph $\mathcal{H}$ is called the Hasse diagram of $K$ [15]. By abuse of language, we will not distinguish between a simplex and a node of $\mathcal{H}$ representing the simplex. Let $\mathcal{H}(i)$ be the nodes of $\mathcal{H}$ corresponding to the $i$-simplicies of $K$. We refer to $\mathcal{H}(i)$ as the level $i$. Each node of $\mathcal{H}$ is equipped with an on/off switch consisting of three colors: red, green, and black. A node colored red means that it is not in the cover $\mathcal{U}$ nor the current collapsible set $U$, a node colored green means that it is in the current collapsible set $U$, and a node colored black means that it is in the cover $\mathcal{U}$. Note that if a node is colored green or black, its red switch must be off. This fact will be used but not stated below. A node can be both black and green. Let $\mathcal{H}_r$ denote the nodes of $\mathcal{H}$ colored red. If $v \in \mathcal{H}(i + 1)$, let $N^i(v)$ be the set of all green nodes on level $i$ connected to $v$ by an edge in $\mathcal{H}$ (i.e. a neighbor of $v$); then $N^i(v) = \{ u \in \mathcal{H}(i) : u$ is green, $u$ is a proper face of $v \}$. Define the expansion set in row $i + 1$ by $E(i + 1) = \{ v \in \mathcal{H}(i + 1) : |N^i(v)| = i + 1 \}$; $E(i + 1)$ collects all $(i + 1)$-simplicies $\sigma$ such that all but one of its proper faces are colored green. The critical expansion set in row $i + 1$ is defined by $CE(i + 1) = \{ v \in E(i + 1) : v$ is red $\}$. We give our algorithm below.
Algorithm 1 Discrete LS category upper bound

Input: A non-empty connected simplicial complex $K$.

Output: A collapsible cover $U$ of $K$.

1. Set $U = \emptyset$ and build the Hasse diagram $H$ of $K$. Color all nodes red.
2. Set $U = \emptyset$.
3. Pick a random red facet $\sigma$ such that $\sigma$ has maximum dimension over all red facets. For every $\tau \subseteq \sigma$ of any dimension, color $\tau$ green.
4. Initialize $i = 0$.
5. If $E(i+1) = \emptyset$, go to step 6. If $CE(i+1) = \emptyset$, choose a random $v \in E(i+1)$. Otherwise, choose a random $v \in CE(i+1)$. Color $v$ (and all $\tau \subseteq v$) and its unique non-green proper face $u$ on level $i$ (and all $\tau \subseteq u$) green. Repeat step 5.
6. Increment $i = i + 1$. If $i = \dim(K)$, go to step 7. Otherwise go to step 5.
7. Add all green nodes to $U$. Color every node in $U$ black and turn off green. Add $U$ to $U$. If $H_r = \emptyset$, then terminate algorithm. Otherwise, go to step 2.

The set $U$ obtained in the above algorithm is a collapsible cover of $K$ so that $\text{dcat}(K) \leq |U| - 1$. Since the complex induced by a facet of $U$ is added in step 3, it follows that it will take at most the number facets of $K$ iterations of the algorithm to find a collapsible cover of $K$, and thus the algorithm will terminate.

The idea behind the algorithm is to determine whether or not an expansion is possible from the information provided by the Hasse diagram. The algorithm begins by picking a random top dimensional subcomplex, and begins to perform elementary expansions by expanding along as many 0-simplicies as possible, as many 1-simplicies as possible, etc. If all of the level $i$ neighbors of a node on level $i+1$ are colored green except one neighbor, this means that all the boundary elements of codimension 1 except one of the corresponding simplex are in the set $U$, and hence we may perform an elementary expansion. A node with color red has not been added to the cover yet, so preference is given to expanding along those nodes on level $i+1$ which are colored red. Since performing a finite number of elementary expansions can be undone by performing elementary collapses in reverse order, the subcomplex we obtain at the end of one full iteration of the algorithm is collapsible. Formally, we have the following.

Proposition 5 The algorithm is correct.

Proof We must show that each $U$ obtained in the algorithm is collapsible and that $\bigcup U = K$. We first show that any $U$ is collapsible by induction. According to step 3, $U = \bar{\sigma}$ which is clearly collapsible. Assume that $U$ is collapsible going into step 5. If $E(i+1) = \emptyset$ and we end up in step 7, then are done. Otherwise, we end up back in step 5 so assume that $E(i+1) \neq \emptyset$ and choose a random $u \in E(i+1)$ or $CE(i+1)$. By definition of these sets, $|N^i(u)| = i + 1$ so that $i + 1$ boundary simplicies of $v$ of dimension $i$ are in $U$ and the $u$ found in step 5
is not in $U$. In other words, $cbd(v) - u \subseteq U$. By Lemma 1, $\{u, v\}$ is a free pair of $U$ so that $U \not\supseteq U \cup \{u, v\}$ is an elementary expansion. Thus $U$ is collapsible. Now let $\sigma \in K$. Since $\sigma$ is in a collapsible $U$ if and only if $\sigma \notin \mathcal{H}_r$ and the algorithm terminates only when $\mathcal{H}_r = \emptyset$, it follows that there exists $U \in \mathcal{U}$ such that $\sigma \in U$. Hence $\bigcup U = K$. □

5 Computation

In this section we present and discuss some experiments we performed on Mathematica using the above algorithm to estimate the discrete category of several well known simplicial complexes. Because the discrete category depends on the particular simplicial structure chosen, the list of facets for these complexes may be found on the Simplicial Complex Library website.\(^2\) In addition, it should be noted that we do not necessarily expect a complex with the topology of a sphere, such as the 3-sphere with a knotted triangle, to have discrete category of 1, nor do we necessarily expect a complex with the topology of a ball, such as the 3-ball with a knotted hole, to have discrete category of 0.

<table>
<thead>
<tr>
<th>Complex</th>
<th>Lower</th>
<th>Upper</th>
<th>Actual category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bing’s house with 2 rooms</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Dunce hat</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3-ball with a knotted hole</td>
<td>0</td>
<td>2</td>
<td>?</td>
</tr>
<tr>
<td>Lockeberg’s 4-polytope</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Mani and Walkup’s 3-sphere</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3-sphere with a knotted triangle</td>
<td>1</td>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>Non-PL 5-sphere</td>
<td>1</td>
<td>7</td>
<td>?</td>
</tr>
<tr>
<td>Poincare sphere</td>
<td>1</td>
<td>8</td>
<td>?</td>
</tr>
<tr>
<td>Projective plane ($\mathbb{R}P^2$)</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Rudin’s 3-ball</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_5$</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$K_{10}$</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$K_{20}$</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>$K_{50}$</td>
<td>24</td>
<td>25</td>
<td>24</td>
</tr>
<tr>
<td>$K_{100}$</td>
<td>49</td>
<td>50</td>
<td>49</td>
</tr>
</tbody>
</table>

It is clear from the table that while our algorithm tends to be accurate for certain complexes, much work still needs to be done. Bing’s house with two rooms [5] and the dunce cap have no free faces (and hence are not collapsible) but have contractible geometric realization. Because these two complexes have no free faces, their discrete category is bounded below by 1, even though Proposition 4 yields a lower bound of 0. In addition, Proposition 2 implies that

\(^2\)http://infoshako.ak.tsukuba.ac.jp/~hachi/math/library/index_eng.html
\( \text{cat}(|K|) \leq \text{dcat}(K) \), where cat is the classical Lusternik–Schnirelmann category. Hence \( \text{cat}(|\mathbb{R}P^2|) = 2 \leq \text{dcat}(\mathbb{R}P^2) \).

We now show that \( 2 \leq \text{dcat}(D) \) where \( D \) is the triangulation of the dunce cap given below.

![Figure 1: A triangulation of the dunce cap](image)

Call any 2-dimensional facet containing 1 2, 1 3, or 2 3 a **boundary facet**.

**Proposition 6** Let \( D \) be the dunce cap given by the triangulation above. Then \( \text{dcat}(D) = 2 \).

**Proof** By the table above, \( \text{dcat}(D) \leq 2 \). Using the above labeling, we show by contradiction that \( \text{dcat}(D) > 1 \), which yields the result. Suppose that \( D = U \cup V \) with \( U, V \) collapsible subcomplexes of \( D \), and assume without loss of generality that every facet of \( U \) and \( V \) is 2-dimensional and that \( U \) and \( V \) do not share any 2-dimensional facets in common. We will utilize the fact discussed leading up to Proposition 4 that a necessary condition for collapsibility is that a complex satisfy Euler’s formula \( v + f - 1 = e \). Since \( D \) is composed of 9 boundary facets, at least one of \( U, V \) must contain 5 such boundary facets, say \( U \). We first claim that if \( U \) contains at least three boundary facets with 1 2, 1 3, and 2 3 in their boundary, then either \( U \) has non-trivial homology or \( U = D \).
Figure 2: If $U$ contains the above boundary facets, then $U$ cannot be collapsible.

The configuration satisfies $5 + 3 - 1 \leq 8$, which implies that the complex is not contractible and hence not collapsible. Every addition of a facet will add either a face and an edge, a face and 2 edges, or a face along with 2 edges and a vertex. In any case, at least one edge is added for every other simplex so that $v + f - 1 = e$ will never be satisfied unless $U = D$, which is not collapsible.

Hence assume that $U$ does not contain 3 boundary facets on the same side. Without loss of generality, suppose that $U$ contains edges 13 and 12, as the other two possibilities are similar. Since $U$ does not contain 23, $V$ contains 238, 237, and 235.

Furthermore, suppose that $U$ contains 136, as a similar analysis shows that if $V$ contains 136, then one of $U, V$ cannot be collapsible. If $U$ also contains 128, then it can be checked that with $U$ and $V$ containing at least the facets mentioned above, the facet 678 will always create a cycle in $U$ and in $V$. 
Figure 3: The minimum collection of facets of $U$ are colored blue and the minimum collection of facets of $V$ are colored green. The addition of any facet with 6 8 to any $U$ or $V$ containing the above blue and green facets, respectively, yields a cycle.

Otherwise, $V$ must contain 1 2 8. Then $U$ contains 1 7 8 for otherwise the addition of 7 8 would create a cycle in $V$. But then as before, edge 6 8 will create a cycle in either $U$ or $V$. Thus $D$ cannot be written as the union of two collapsible sets and $1 < \text{dcat}(D)$ which is what we desired to show. \hfill \Box

Remark 7 As an alternative to show that $\text{dcat}(D) \leq 2$, one could use the discrete Lusternik–Schnirelmann theorem \cite{1} which says that if $f: K \to \mathbb{R}$ is a discrete Morse function with $m$ critical values, then $\text{dcat}(K) + 1 \leq m$. There is a discrete Morse function $g: D \to \mathbb{R}$ with exactly 3 critical values \cite{2} so that by the discrete LS theorem, $\text{dcat}(D) + 1 \leq 3$. As computed above, $\text{dcat}(D) = 2$ so this provides an example of equality in the discrete LS theorem.

Although the above evidence does not warrant a conjecture, Proposition 6 suggests that the existence of a contractible complex with discrete category any positive integer is worth investigating; that is, given a positive integer $n$, does there exist a contractible simplicial complex $A(n)$ such that $\text{dcat}(A(n)) = n$? Bing’s House with two rooms and the dunce cap answer the question in the affirmative for $n = 1$ and $n = 2$, respectively.

References

\begin{itemize}
\end{itemize}


