0. Multiple integrals using Derive: Examples

0.1 Double integrals

\[ \iint_D f(x, y) \, dx \, dy = \text{INT(\text{INT}(f(x, y), y, c, d), x, a, b)} \]

**Example 1:** Find the integral of \( f(x, y) = x^3y^2 + xy \) over the rectangle \( R: 0 \leq x \leq 1 \) and \( 1 \leq y \leq 2 \).

The integral can be computed by evaluating one of the following iterated integrals:

\[ \int_1^0 \int_1^2 f(x, y) \, dx \, dy \quad \text{or} \quad \int_1^0 \int_1^2 f(x, y) \, dy \, dx \]

First, define the function \( f \) by authoring the expression (using the pencil icon) or using "Declare/Function" from the menu:

\[ f(x, y) := x^3y^2 + xy \]

To get \( \int_1^0 \int_1^2 f(x, y) \, dx \, dy \), author (using the pencil icon) the expression \( \text{INT(\text{INT}(f(x, y), y, 1, 2), x, 0, 1)} \). Click on \( \text{OK} \), then press the \( \approx \) button (or \( \approx \) for an approximation). You will get \( \Rightarrow 4/3 \).

Repeat the process with the expression \( \text{INT(\text{INT}(f(x, y), x, 0, 1), y, 1, 2)} \) to get \( \int_1^2 \int_0^1 f(x, y) \, dy \, dx \).

**Example 2:** Find the volume of the solid which lies under the surface \( z = xy^2 \) over the triangular region described by \( 0 \leq x \leq 1 \) and \( x \leq y \leq 2x \).

- You must plot the region with "x between 0 and 1, and between the curves \( y = x \) and \( y = 2x\)" in order to find the limits of integrations.

- Using Derive, author the expression:

\[ [x = 0, x = 1, y = x, y = 2x] \]

Then press the button \( \text{→} \) to create the graphs in the 2-D Plot window, and use the "zoom buttons", the "center icons" or the "set-range icons" to have a good view of the region.

- The volume is computed then by evaluating the integral \( \int_0^1 \int_x^{2x} xy^2 \, dy \, dx \).

- So, simply author the expression \( \text{INT(\text{INT}(xy^2, y, x, 2x), x, 0, 1)} \), and click on \( \text{OK} \), then on \( \approx \). You will get \( \Rightarrow 7/15 \).

**Example 3:** Find the volume of the solid that lies under the surface \( z = x^2y^2 \) over the region described by \( 0 \leq x \leq \sin(y) \) and \( 0 \leq y \leq \pi \).

- You must plot the region "\( x \) between the curves \( x = 0 \) and \( x = \sin(y) \), and \( y \) between 0 and \( \pi \)". So, author the expression:

\[ [x = 0, x = \sin(y), y = 0, y = \pi] \]

- Then proceed like in example 2. The volume is computed by evaluating the integral \( \int_0^\pi \int_0^{\sin(y)} x^2y^2 \, dx \, dy \) by authoring the expression \( \text{INT(\text{INT}(x^2y^2, x, 0, \sin(y)), y, 0, \pi)} \); you get \( \square \).
0.2 Polar coordinates

\[ \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r \cos \theta, r \sin \theta) \ r \ dr \ d\theta \]

with \( x = r \cos \theta, y = r \sin \theta, x^2 + y^2 = r^2 \)

Example 4: Find the integral of \( f(x, y) = x - y^2 \) over the sector of the unit disk in which \( -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \).

- In this problem, we have \( 0 \leq r \leq 1 \) and \( -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \). So the integral is \( \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{1} f(r \cos \theta, r \sin \theta) \ r \ dr \ d\theta \).

- The computation is performed by first defining the function \( f \), like in example 1, and then authoring the expression (using \( t \) instead of \( \theta \)):

  \[
  \text{INT} \left( \text{INT} \left( f(r \cos(t), r \sin(t)) \cdot r, r, 0, 1 \right), t, -\frac{\pi}{4}, \frac{\pi}{4} \right)
  \]

- Important: Do not forget to multiply \( f \) by \( r \). Then press the \( \equiv \) icon. You will get \( \boxed{\text{answer}} \).

Example 5: Compute the volume of the solid that lies under the surface \( z = x + y^2 \) and above the disk bounded by the circle in \( x^2 + (y - 1)^2 = 1 \).

- You must plot this boundary disk: author \( x^2 + (y - 1)^2 = 1 \) and plot it.

- You need a polar-coordinates description of this disk, this is to say we need the bounds on \( r \) and \( \theta \).

- First, use the substitution \( x = r \cos t, y = r \sin t \) in the equation of the boundary (the circle). Note that we are using \( t \) instead of \( \theta \) in derive for simplification. So, author:

  \[
  (r \cos t)^2 + (r \sin t - 1)^2 = 1
  \]

  Then simplify and use the \text{SOLVE} \ command to determine \( r \). You get \( \left[r = 0, r = 2 \sin(t)\right] \).

- Consequently, the disk is described by \( 0 \leq r \leq 2 \sin(t) \) and \( 0 \leq \theta = t \leq \pi \) (see the graph of the disk to understand why \( 0 \leq \theta \leq \pi \)).

- The volume is then given by the integral \( \int_{0}^{\pi} \int_{0}^{2 \sin t} f(r \cos t, r \sin t) \ r \ dr \ dt \) where \( f(x, y) = x + y^2 \)

- Like in example 4, the computation is performed by first defining the function \( f \), and then authoring the expression

  \[
  \text{INT} \left( \text{INT} \left( f(r \cos(t), r \sin(t)) \cdot r, r, 0, 2 \sin(t) \right), t, 0, \pi \right)
  \]

  Then press \( \text{OK} \) and \( \equiv \).

You get \( \boxed{\frac{5 \pi}{4}} \).

1. Below is the lab-problems due next Tuesday

Your report should include only this Part. Each Lab group should turn in only one report. Number and label all answers with necessary explanations and Graphs. Do not hand in unnecessary work! Your Lab report is supposed to be well organized and with all the explanations needed. Each problem worths 5 points and the quality of the presentation of the report worths 5 points.

How to save you Algebra work as a Microsoft WORD-FILE: This is a very practical way for typing and presenting your lab-report. First, in the Word file, select Format/Font and set the font to be DfW Printer, then simply “Copy” the work from the DERIVE Algebra window, and “Paste” it on the Word file. The trick of changing the font is necessary otherwise it will not look nice.

Problem 1. # 30 section 15.3
Problem 2. # 33 section 15.3
Problem 3. # 16 section 15.4
Problem 4. # 26 section 15.4