Lab #3: Limits Graphically, Numerically & using Derive: Application to graph sketching

1. GOALS OF THIS LAB

The goals are to understand the concept of the limit of a function both graphically and numerically, to find the limit using a computer algebra system, to find the vertical and horizontal asymptotes of a curve, and to use these to sketch the graph of a function in the best viewing window showing all its important aspect.

2. IMPORTANT REMARKS

Work with your partner(s). Each Lab group should turn in only one report for grading. NUMBER and LABEL all answers and all Graphs. DO NOT HAND IN UNNECESSARY WORK! Where appropriate, you must explain your reasoning in text (type it). It is not acceptable to just staple together a bunch of math formulas and some computer graphics printouts. Your Lab report is supposed to be well organized and with all the explanations needed. The lab-report will be graded on its quality and its presentation. See Guide to Lab at my the web page.

3. HELP FOR THIS LAB

- The Lab #1 took you through most (but not all) of the DERIVE commands you need. So, have it with you during this lab to review the commands needed. It is posted on the HOMEWORK web page.

- In particular, review in lab#1 how to make a table of values from a given function, and how to save your work as a Microsoft Word file to improve the presentation of your lab report.

- I strongly recommend that you save as a Word-file your work and open a new DERIVE file after each problem.

- Before starting each of the problems assigned below, it is important to look at similar examples in sections 2.2, 2.3 and 2.6 of the Textbook as well as the corresponding concepts.

- The Part #0 describes in details how to use the graphical and numerical techniques in DERIVE to find limits of a function. [DO NOT include] this Part 0 with your Lab Report. But you have to do it first to be familiar with these techniques in order to do well in this Lab.

- Below is a reminder of all the definitions and notations related to the concept of “limit”. It is included to help you in your work.
4. Reminder about limits

(1) $\lim_{x \to a^-} f(x) = L$, if we can make the values of $f(x)$ as close to $L$ as we like by taking $x$ to be sufficiently close to $a$ from the left (so $x < a$), but not equal to $a$.

(2) $\lim_{x \to a^+} f(x) = L$, if we can make the values of $f(x)$ as close to $L$ as we like by taking $x$ to be sufficiently close to $a$ from the right (so $x > a$), but not equal to $a$.

(3) $\lim_{x \to a} f(x) = L$, if we can make the values of $f(x)$ as close to $L$ as we like by taking $x$ to be sufficiently close to $a$ from both sides, but not equal to $a$.

■ Existence of the limit:

(4) If this $L = +\infty$, $L = -\infty$ or does not exist, we say that the limit does not exist.

(5) In particular, $\lim_{x \to a} f(x)$ exists $\iff$ $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \neq \mp \infty$

■ Vertical Asymptote:

(6) $\lim_{x \to a} f(x) = +\infty$, if we can make the values of $f(x)$ as large & positive as we like by taking $x$ to be sufficiently close to $a$ from both sides, but not equal to $a$.

(7) $\lim_{x \to a} f(x) = -\infty$, if we can make the values of $f(x)$ as large & negative as we like by taking $x$ to be sufficiently close to $a$ from both sides, but not equal to $a$.

(8) In a similar way we define $\lim_{x \to a^-} f(x) = \mp \infty$ and $\lim_{x \to a^+} f(x) = \mp \infty$.

(9) The vertical line $x = a$ is called vertical asymptote of the curve $f(x)$ if at least one of the above limits is $+\infty$ or $-\infty$. (See Figures 11-17 at pages 99-101 of the textbook)

■ Horizontal Asymptote:

(10) $\lim_{x \to +\infty} f(x) = L$, if we can make the values of $f(x)$ as close to $L$ as we like by taking $x$ sufficiently large (so, close to $+\infty$).

(11) $\lim_{x \to -\infty} f(x) = L$, if we can make the values of $f(x)$ as close to $L$ as we like by taking $x$ sufficiently large negative (so, close to $-\infty$).

(12) The horizontal line $y = L$ is called horizontal asymptote of the curve $f(x)$ if either $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$. (See figures 2-7 at pages 136-139 of the textbook)
PART 0

To find \( \lim_{x \to a} f(x) \), you will need to investigate the values \( f(x) \) as \( x \) moves close to the value \( a \) (from both sides) graphically, numerically and directly from Derive.

Practice first with this example to see if \( \lim_{x \to 0} \frac{\sin x}{x} \) exists and find its value \( L \).

(1) Graphically:

- **Declare** the function \( f(x) := \frac{\sin x}{x} \) and Plot its graph. (Remember to split the screen into two windows!)
- Choose **Options/Trace Mode** from the menu. The cross "+" is then displayed as a small square "□" that moves along the graph. Therefore the coordinates of the position of this square are in the form \((x, f(x))\) that you can read at the bottom left of the plot window.
- Move this "□" slowly toward \( x = 0 \) using the left and right arrow keys, and read the values of the coordinates.
- What can you say about \( \lim_{x \to 0^-} f(x) \)? Why? (See Reminder #1 on page 2)
- What can you say about \( \lim_{x \to 0^+} f(x) \)? Why? (See Reminder #2 on page 2)
- What can you say about \( \lim_{x \to 0} f(x) \)? Why? (See Reminder #3 on page 2)

(2) Numerically:

- You need to make a table (or tables) of values with 2 columns: the first column of several values of \( x \), where \( x \) is getting closer and closer to 0 from the right and then from the left, and the second column with the corresponding values of \( f(x) \). For example: for \( x = -0.5, -0.3, -0.2, -0.1, -0.01, -0.001, \) and \( x = 0.5, 0.3, 0.2, 0.1, 0.01, 0.001 \). (See example page 95 of the textbook)
- To do that, Declare the function, then use the matrix \( [::] \) button and fill out the form: choose the number of columns and rows, then enter the values of \( x \) in the first column and their corresponding \( f(x) \) values in the second column; for example in the first row enter \(-0.5\) and \( f(-0.5) \). See ”Make table of values from a function” from Lab #1.
- What can you say about \( \lim_{x \to 0^-} f(x) \)? Why? (See Reminder #1 on page 2)
- What can you say about \( \lim_{x \to 0^+} f(x) \)? Why? (See Reminder #2 on page 2)
- What can you say about \( \lim_{x \to 0} f(x) \)? Why? (See Reminder #3 on page 2)

(3) Directly from Derive

Click the \( \lim \) button and fill in the form. Click \( \text{OK} \), then click on \( = \). Use it to calculate \( \lim_{x \to 0^-} f(x) \), \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0} f(x) \). Compare them to your previous answers.

DO NOT INCLUDE THIS PART 0 IN YOUR LAB REPORT, IT IS HERE TO PRACTICE THESE LIMIT-TECHNIQUES. YOU LAB REPORT SHOULD INCLUDE THE FOLLOWING PROBLEMS:
5. Lab Report Requirement

For each of the following problems, include any needed graph(s) in your report with all necessary comments and explanations. I recommend that you copy the graphs and paste them in a “Word Document” so you can easily save your work and include any comments. You do not need to include all DERIVE manipulations. See Lab #0 and page 1 of this lab for more details.

**Problem 1: Limits Graphically and Numerically**

Use the three techniques described in PART 0 to investigate \( \lim_{x \to 0} f(x) \) where \( f(x) = (1 + x)^{\frac{1}{x}} \).

**Problem 2: Vertical Asymptotes**

Use only the **Graphical** technique, to investigate \( \lim_{x \to 5} g(x) \) where \( g(x) = \frac{6}{x - 5} \).

1. What can you say about \( \lim_{x \to 5^-} \frac{6}{x - 5} \)? Why? (See Reminders #6, 7 & 8 on page 2)
2. What can you say about \( \lim_{x \to 5^+} \frac{6}{x - 5} \)? Why? (See Reminders #6, 7 & 8 on page 2)
3. What can you say about \( \lim_{x \to 5} \frac{6}{x - 5} \)? Why? (See Reminders #4 & 5 on page 2)
4. Confirm your answers using the \( \lim \) button.
5. Do we have vertical asymptotes? (See Reminder #9 on page 2)

**Problem 3: Limits at infinity and Horizontal Asymptotes**

1. Use **only** the **Graphical** technique (described in PART 0) to investigate (See Reminders #10 & 11 on page 2): \( \lim_{x \to +\infty} h(x) \) where \( h(x) = (x^3 + x^2 + 1)^{\frac{1}{x^3 - x}} \).
2. Check your answer using the \( \lim \) button.
3. Do you have any horizontal asymptote at infinity? why?

**Problem 4: Graphing a function with all its important aspects**

Consider the function \( f(x) = \frac{x^3 - 5x + 2}{x^3 - 8x^2 + 2x + 20} \).

1. Find the exact zeros (not an approximation) of the denominator using Derive.
2. Using the \( \lim \) button, calculate \( \lim f(x) \) at each of the values where \( f \) is undefined.
3. Use the previous question to find all the vertical asymptotes and justify your answer.
4. Do all the zeros you found in question (1) appear as vertical asymptotes? If some are missing explained what happened. **Hint**: Derive can factor out; see Lab#1.
5. Find all the horizontal asymptotes. (See Reminder #12 on page 2)
6. Plot the graph of \( f \) with all its asymptotes and all necessary labels. (See Reminders #9 & #12 on page 2 for how to write the equations of the vertical and horizontal asymptotes)