To receive credit for your answers you must show all your work, explain your reasoning carefully and clearly, and include all steps necessary to completely justify each answer. Any variables you use must be clearly identified. [Box] in your answers when it is possible. Each problem worths 20 points. Good luck!

Problem 1. Find the particular solution of dumped harmonic oscillator: \( \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 5y = 0 \), with the initial conditions: \( y(0) = 1, y'(0) = 1 \), and describe its behavior.

The characteristic equation: \( \lambda^2 + 4\lambda + 5 = 0 \)
\[ \Delta = 4^2 - 4(5) = -4 \implies \lambda = -2 \pm i \sqrt{2} \]

Thus, the general solution is: \( Y = e^{-2t}[k_1 \cos t + k_2 \sin t] \)

Since \( y(0) = 1 \), then \( e^0[k_1 + 0] = k_1 = 1 \)

On the other hand, \( y'(t) = -2 e^{-2t} [\cos t + k_2 \sin t] + e^{-2t} [-\sin t + k_2 \cos t] \)

In particular, \( y'(0) = -2 [1 + 0] + [0 + k_2] = -2 + k_2 \)

Since \( y'(0) = 1 \), then \( 1 = -2 + k_2 \) \( \implies k_2 = 3 \)

Therefore: \( y(t) = e^{-2t} [\cos t + 3 \sin t] \)

The solution curve will spiral toward the equi. pt \((0,0)\) with a period of \( \frac{2\pi}{3} = \frac{2\pi}{3} \) and in a clockwise direction.
**Problem 2.** Consider the system: \( \frac{dY}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} Y, \) where \( Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \).

1. Find the equilibrium points, their type and the eigenvalues.
2. Give a formula for the general solution.
3. Sketch the phase portrait with some representative solutions (two in each region).
4. Find a formula for the solution with initial condition \( Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

**Step 1:**
\[ \det A = 4 - 1 = 3 \neq 0 \implies \text{only } (0,0) \text{ as equ. pt.} \]
\[ \det A_\lambda = (-2-\lambda)(-2-\lambda) - 1 = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) \]
Thus eigenvalues: \( \lambda = -3 \) \& \( \lambda = -1 \) \( \implies \text{Real sink} \).

**Step 2:**
*Eigenvector for } \lambda = -3:*
\[ AY = -3Y \implies -2x + y = -3x \implies \begin{pmatrix} y = -x \\ y = -x \end{pmatrix} \]
with \( x = 1 \) --- one eigenvector is \[ V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

*Eigenvector for } \lambda = -1:*
\[ AY = -1Y \implies -2x + y = -x \implies \begin{pmatrix} y = x \\ y = x \end{pmatrix} \]
--- eigenvector is \[ V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

So, a general solution:
\[ Y = k_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

**Step 3:**

**Step 4:**
\[ Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
\[ \implies \text{which is the equilibrium pt } (0,0) \]
Thus
\[ Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
for all \( t \).
Problem 3. Consider the system \( \frac{dY}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} Y \)

1. Find the eigenvalues and its corresponding eigenvectors, and the straight-line solutions.
2. Sketch the phase plane with some representative solutions.
3. Sketch the graph of \( x(t) \) and \( y(t) \), where \( Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \) is the solution with initial condition \( Y(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \).

\[ \det A = (\lambda - 2)(\lambda - 4) + 1 = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2. \]

\( \Rightarrow \) Repeated Eigenvalue \( \lambda = -3 \) with eigenvector \( V = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

because \( AY = -3Y \Rightarrow -2x - y = -3x \Rightarrow x = y \Rightarrow Y = \begin{pmatrix} x \\ x \end{pmatrix} \)

\( \Rightarrow \) only one straight line: \( Y_1 = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

\[ \text{At } (1, 1), \quad \frac{dY}{dt} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \]

\( \Rightarrow \) Counter-clockwise.
Problem 4. Consider the system: \( \frac{dY}{dt} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} Y \), where \( Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \).

1. Find the the eigenvalues (NOT the eigenvectors).

2. Describe in words the general behavior of the solution curves (direction, period, long term behavior, where the solution curves are coming from, etc.)

3. Sketch the graph of the corresponding solutions \( x(t) \) and \( y(t) \) for the initial condition \( Y(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \).

\[ \text{det} \, A = (3-\lambda)(1-\lambda) - 2 = \lambda^2 - 4 \lambda + 1 \]
\[ \lambda = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3} \quad \begin{cases} \lambda_1 = 2 - \sqrt{3} \\ \lambda_2 = 2 + \sqrt{3} \end{cases} \]

- Both eigenvalues are positive real numbers, therefore most of the solution curves come from \((0,0)\), tangent to the straight line corresponding to \( \lambda_1 = 2 - \sqrt{3} \) and tend away from to \((0,0)\) parallel to the straight line corresponding to \( \lambda_2 = 2 + \sqrt{3} \).

Special cases:

- Only solutions starting at \((0,0)\) stay at \((0,0)\).
- Solutions starting at one of the straight lines, tend away from \((0,0)\) along the same straight line.

Find eigenvectors, Draw straight line and use phase portrait to find the corresponding graphs of \( x(t) \) \& \( y(t) \).
Problem 4, second version: Consider the system: \( \frac{dY}{dt} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} Y \), where \( Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \).

1. Find the eigenvalues (NOT the eigenvectors).

2. Describe in words the general behavior of the solution curves (direction, period, long term behavior, where the solution curves are coming from, etc.)

3. Sketch the graph of the corresponding solutions \( x(t) \) and \( y(t) \) for the initial condition \( Y(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \).

\[ \text{det} A = \lambda^2 - 4\lambda + 5 \implies \lambda = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = \frac{2}{2} \pm i \]

Except solutions starting at \((0, 0)\), all solution curves spiral away from \((0, 0)\), tend to infinity, with period \( \frac{2\pi}{\beta} \), and in a counterclockwise direction. because at the point \((1, 0)\), \( \frac{dY}{dt} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \) is a vector that leads to counterclockwise direction.
Problem 5. Find all the bifurcation values of the parameter \( a \) in the following family of differential systems, sketch the curve that represents the system in the trace-determinant plane, and briefly describe the behaviors exhibited by the system as \( a \) increases.

\[
\begin{align*}
\frac{dy}{dt} &= \begin{pmatrix} 2a & -1 \\ a^2 + 4 & -a \end{pmatrix} y & -\infty < a < \infty \\
T &= 2a - a = a \\
D &= -2a^2 + a^2 + 4 = 4 - a^2 \\
\text{Thus,} \quad D &= 4 - T^2 \rightarrow \text{Parabola opening downwards}
\end{align*}
\]

There are 5 possible locations for bifurcations:

- \( A \& F \): Intersection with \( t \)-axis \( D = 0 \) \( \Rightarrow 4 - a^2 = 0 \) \( \Rightarrow a = \pm 2 \)
- \( C \): \( D \)-axis \( T = 0 \) \( \Rightarrow a = 0 \)
- \( B \& E \): \( D = \frac{T^2}{4} \) \( \Rightarrow 4 - a^2 = \frac{a^2}{4} \) \( \Rightarrow 16 - 4a^2 = a^2 \) \( a = \pm \frac{4}{\sqrt{5}} \)

\[
\begin{align*}
a &< -2 \rightarrow \text{Saddle} \\
a &= -2 \rightarrow \text{Bifurcation} \\
-2 < a < -\frac{4}{\sqrt{5}} \rightarrow \text{Real sink} \quad \text{(between A \& B)} \\
a &= -\frac{4}{\sqrt{5}} \rightarrow \text{Bifurcation} \\
-\frac{4}{\sqrt{5}} < a < 0 \rightarrow \text{Spiral sink} \quad \text{(between B \& C)} \\
a &= 0 \rightarrow \text{Bifurcation} \\
0 < a < \frac{4}{\sqrt{5}} \rightarrow \text{Spiral source} \quad \text{(between B \& E)} \\
a = \frac{4}{\sqrt{5}} \rightarrow \text{Bifurcation} \\
\frac{4}{\sqrt{5}} < a < 2 \rightarrow \text{Real source} \quad \text{(between E \& F)} \\
a = 2 \rightarrow \text{Bifurcation} \\
a > 2 \rightarrow \text{Saddle}
\end{align*}
\]