Solution of TEST 5: Math 211-Multivariate Calculus Spring 2003

Problem 1. Using the appropriate coordinates, SET UP, but do not calculate, the triple integral \[ \iiint_E \sqrt{x^2 + y^2 + z^2} \, dV \]
where \( E \) is the upper half of the sphere of center \((0,0,0)\) and radius 1.

- **Solution:** With the spherical coordinates we have \( x^2 + y^2 + z^2 = \rho^2 \) and \( dV = \rho^2 \sin \phi d\rho d\theta d\phi \).
  
  Since the domain is the upper half of the sphere with radius 1, then it can be described by \( 0 \leq \rho \leq 1, \, 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \frac{\pi}{2} \) because it is only the upper half. Therefore
  
  \[ \iiint_E \sqrt{x^2 + y^2 + z^2} \, dV = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \rho^3 \sin \phi \, d\rho d\theta d\phi \]

Problem 2. Let \( E \) be the region inside the paraboloid \( z = 3x^2 - 3y^2 = 0 \), which is bounded above by the paraboloid \( z = 16 - x^2 - y^2 \). Use cylindrical coordinates to SET UP, but do not calculate, the triple integral that gives the volume of this region \( E \).

- **Solution:** With the cylindrical coordinates we have \( x^2 + y^2 = \rho^2 \) and \( dV = \rho \, dz \, d\rho \, d\theta \).
  
  So, in the region \( E \), \( z \) is above the paraboloid \( z = 3x^2 + 3y^2 = 3\rho^2 \) and below the paraboloid \( z = 16 - (x^2 + y^2) = 16 - \rho^2 \).
  
  To find the corresponding values of \( \rho \), we have the find the intersection of the two surfaces above: \( 3\rho^2 = 16 - \rho^2 \Rightarrow 4\rho^2 = 16 \Rightarrow \rho = 2 \). So, inside the circle of radius 2, therefore \( 0 \leq \rho \leq 2 \) and \( 0 \leq \theta \leq 2\pi \).

Thus the volume of \( E \) is
  
  \[ V = \iiint_E 1 \, dV = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} \int_{z=3\rho^2}^{z=16-\rho^2} \rho \, dz \, d\rho \, d\theta \]

Problem 3. Use an appropriate change of variables to evaluate \( I = \int_R (6x - 3y) \, dA \), where \( R \) is the region bounded by the lines \( 2x - y = 1 \), \( 2x = 3 + y \), \( x + y = 1 \) and \( x + y = 2 \).

- **Solution:** The equation of these lines are are equivalent to \( 2x - y = 1 \), \( 2x = 3 + y \), \( x + y = 1 \) and \( x + y = 2 \). Therefore it is natural to choose:
  
  \[ \begin{align*}
  u &= 2x - y \\
  v &= x + y
  \end{align*} \]

  So
  
  \[ \begin{align*}
  1 \leq u &\leq 3 \text{ and } 1 \leq v \leq 2 \\
  u + v &= 3x \Rightarrow x = \frac{u + v}{3} \Rightarrow J = \left| \begin{array}{cc}
  \frac{px}{\partial u} & \frac{px}{\partial v} \\
  \frac{py}{\partial u} & \frac{py}{\partial v}
  \end{array} \right| = \left| \begin{array}{cc}
  \frac{1}{3} & \frac{1}{3} \\
  \frac{1}{3} & \frac{1}{3}
  \end{array} \right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}
  \end{align*} \]

  Thus
  
  \[ I = \int_{R} (3u - 2v) \, dA = \int_{v=1}^{v=2} \int_{u=1}^{u=3} \frac{1}{3} u \, du \, dv = 4 \]

Problem 4. Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F}(x,y,z) = x \mathbf{i} + yz \mathbf{j} + z^2 \mathbf{k} \), and \( C \) consists of two curves \( C_1 \) and \( C_2 \) with: \( C_1 \) is the line segment from \((0,0,0)\) to \((2,0,0)\), and \( C_2 \) is a curve from the point \((2,0,0)\) to \((0,4,2)\) which is represented by the vector function \( \mathbf{r}(t) = (2-t) \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k} \) for \( 0 \leq t \leq 2 \).

- **Solution:** \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \). For each integral we have to express everything in terms of \( t \)

  \( C_1 \) is the line segment from \( A = (0,0,0) \) to \( B = (2,0,0) \). So a representation of \( C_1 \) is given by

  \[ r(t) = A + t \overrightarrow{AB} = (0 + t(2 - 0), 0 + t(0 - 0), 0 + t(0 - 0)) = (2t, 0, 0) \text{ for } 0 \leq t \leq 1 \]

  So, \( dr = r'(t) \, dt = (2, 0, 0), \) and \( F = (x, yz, z^2) \Rightarrow F(r(t)) = (2t, 0, 0) \).

  Thus
  
  \[ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{t=1} (2t, 0, 0) \cdot (2, 0, 0) \, dt = \int_{0}^{1} 4 \, dt = [2t^2]^1_0 = 2 \]

  \( C_2 \) is represented by the vector function \( \mathbf{r}(t) = ((2-t), t^2, t) \) for \( 0 \leq t \leq 2 \).

  So, \( dr = r'(t) \, dt = (-1, 2t, 1), \) and \( F = (x, yz, z^2) \Rightarrow F(r(t)) = (2-t, 2t^2, t^2) \).

  Thus
  
  \[ \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{t=2} (2-t, t^2, t^2) \cdot (-1, 2t, 1) \, dt = \int_{0}^{2} (-2 + t + 2t^4 + t^2) \, dt = [-2t + \frac{t^2}{2} + \frac{2t^5}{5} + \frac{t^3}{3}]^2_0 = 13.46 \]

**Conclusion:** \( \int_C \mathbf{F} \cdot d\mathbf{r} = 2 + 13.46 = 15.46 \)
Problem 5. Consider the vector field: \( F(x, y) = 3x^2y^2 \mathbf{i} + 2x^3y \mathbf{j} \).

1. Show that \( F \) is a conservative vector field.
2. Find a function \( f(x, y) \) such that \( F = \nabla f \).
3. Evaluate the line integral \( \int_C F \cdot d\mathbf{r} \) where \( C \) is any curve from \((1, 2)\) to \((2, 1)\).

- **Solution:** With \( P = 3x^2y^2 \) and \( Q = 2x^3y \), then \( F = P \mathbf{i} + Q \mathbf{j} \).

1. \( \frac{\partial P}{\partial y} = 6x^2y \) and \( \frac{\partial Q}{\partial x} = 6x^2y \). So \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \implies F \) is a conservative vector field.

2. \( \nabla f = F \implies \frac{\partial f}{\partial x} = 3x^2y^2 \) and \( \frac{\partial f}{\partial y} = 2x^3y \).

3. By the Fundamental Theorem for Line Integrals \( \int_C F \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(r(b)) - f(r(a)) \) with \( r(a) = (1, 2) \) is the starting point of the curve and to \( r(b) = (2, 1) \) is the ending point of the curve.

Thus \( \int_C F \cdot d\mathbf{r} = f(2, 1) - f(1, 2) = 2^3 - 1^3 = 2 = 4 \)

Problem 6. Evaluate \( I = \oint_C [ x \cos x + e^{\sin x} + 3y ] \, dx + [ 5x + (1 + y^2)^3 + \cos (e^y + y^2) ] \, dy \) where \( C \) is the positively-oriented boundary of the upper half disk \( D \) of radius 2 and center \((0, 0)\).

- **Solution:** With \( P = [ x \cos x + e^{\sin x} + 3y ] \) and \( Q = [ 5x + (1 + y^2)^3 + \cos (e^y + y^2) ] \), and since \( C \) is a simple closed positively oriented curve, then by Green’s Theorem we have:

\[
\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_D (5 - 3) \, dA = 2 \iint_D dA = 2 \text{[area of half disk of radius 2]} = 4\pi
\]