Problem 1. Write the equation \( r = 6 \cos \theta \) in rectangular coordinates and identify completely the corresponding surface (describe in words the surface, axis, center, etc.)

Solution: Since \( x = r \cos \theta \) then \( \cos \theta = \frac{x}{r} \). Thus, by substitution the equation \( r = 6 \cos \theta \) becomes \( r = 6 \frac{x}{r} \), or again \( r^2 = 6x \). Since \( r^2 = x^2 + y^2 \) we then have: \( x^2 + y^2 = 6x \iff x^2 + y^2 - 6x = 0 \iff (x - 3)^2 + y^2 = 3^2 \), which is the equation of a circular vertical cylinder with axis parallel to the \( z \)-axis and passing through the point \((3,0,0)\).

Problem 2. Match the equations with their graphs. Give reasons for your choices to receive credit.

(a) \( 8x + 2y + 3z = 0 \): Obviously it is the equation of a plane, thus it corresponds to the surface \( \text{II} \).

(b) \( z = e^x \): Since \( x \) is missing it is then a cylinder with the main curve \( z = e^x \) on the \( yz \)-plane shifted along the \( x \)-axis, thus it corresponds to the surface \( \text{III} \).

(c) \( z = \sin x + \cos y \): The vertical traces parallel to the \( xz \)-plane (resp. \( yz \)-plane) are in the shape of a sine function (resp. cosine), thus it corresponds to the surface \( \text{I} \).

(d) \( z = \sin \left( \frac{\pi}{2 + x^2 + y^2} \right) \): Clearly it corresponds the the remaining surface \( \text{IV} \) which makes sense because for \( x \) and \( y \) large, the denominator is too large, thus \( z \approx \sin(0) = 0 \).

Problem 3. Consider the vector function \( r(t) \) describing the curve shown below. Put the curvatures of \( r \) at \( A, B \) and \( C \) in order from the smallest to largest. Justify your answer to receive credit.

Solution: By graphing the osculating circle (a circle tangent to the curve at each of the 3 points, it is clear that the circle at \( B \) has the largest radius and that the one in \( C \) has the smallest radius. Therefore, since \( R = \frac{1}{K} \) where \( K \) is the curvature, it follows that the curvature at \( B < \text{curvature at \( A < \text{curvature at \( C \).} \)

Problem 4. Show that the space curve given by \( r(t) = < t \sin(5t), t \cos(5t), t^2 > \) lies in a known surface, and give the equation and a description of this surface.

Solution: We have \( z = t \sin(5t) \), \( y = t \cos(5t) \) and \( z = t^2 \). So in particular \( x^2 + y^2 = t^2(\sin^2(5t) + \cos^2(5t)) = t^2 = z \). Thus the curve lies in the surface \( x^2 + y^2 = z \) which is a circular paraboloid opening upward with axis the \( z \)-axis.

Problem 5. Give a parametric representation of the intersection of the paraboloid \( z = x^2 + y^2 \) and the plane \( z - 2y = 0 \). (Hint: completing the square)

Solution: \( z - 2y = 0 \iff z = 2y \), then by substitution in the equation of paraboloid:

\[
    z = x^2 + y^2 \Rightarrow 2y = x^2 + y^2 \Leftrightarrow x^2 + y^2 - 2y = 0 \Leftrightarrow x^2 + (y - 1)^2 = 1
\]

Which a circle in the \( xy \)-plane. Put then \( x = \cos(t) \) and \( (y-1) = \sin(t) \), i.e. \( y = 1 + \sin(t) \).

Finally \( z = 2y = 2 + 2\sin(t) \).

Problem 6. Consider the vector function \( r(t) = < \sin(2t), t, \cos(2t) > \) and the point \( P(0, \pi, 1) \) of the corresponding space-curve.

(1) Describe (in words) this curve.

It is a horizontal clockwise "HELIX" with axis the \( y \)-axis and radius 1

(2) Find an equation of the tangent line at the point \( P \). First, the point \( P \) corresponds to the time \( t = \pi \).

The derivative of \( r(t) \) is \( r'(t) = < 2 \cos(2t), 1, -2 \sin(2t) > \)

thus, at the point \( P, r(\pi) = < 2 \cos(2\pi), 1, -2 \sin(2\pi) > = < 2, 1, 0 > \)

Thus, the equation of the tangent line is: \( L(t) = r'(\pi) \cdot t + OP \) \( \Rightarrow \) \( t < 2t, \pi, 1 > \).

(3) Find the length of the curve from the point \( Q(0,0,1) \) to the given point \( P \).

\[
    \int_0^\pi \sqrt{(2\cos(2t))^2 + 1^2 + (-2\sin(2t))^2} = \sqrt{4(\cos(2t))^2 + (\sin(2t))^2 + 1} = \sqrt{4 + 1} = \sqrt{5}.
\]

And the point \( Q(0,0,1) \) corresponds to \( t = 0 \). Thus the length is given by:

\[
    L_{QP} = \int_0^\pi |r'(t)| dt = \int_0^\pi \sqrt{5}dt = [\sqrt{5}t]^\pi_0 = \sqrt{5} \pi.
\]

(4) Find the vectors \( T, N, \) and \( B \) at the point \( P \).

Again, the point \( P \) corresponds to \( t = \pi \).

\[
    T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{5}} < 2 \cos(2t), 1, -2 \sin(2t) > \Rightarrow T(\pi) = \frac{1}{\sqrt{5}} < 2, 1, 0 >
\]

\[
    T'(t) = \frac{1}{\sqrt{5}} < -4 \sin(2t), 0, -4 \cos(2t) > \Rightarrow T'(\pi) = \frac{1}{\sqrt{5}} < 0, 0, -4 >, \text{ and } |T'(\pi)| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2} = \frac{4}{\sqrt{5}}
\]

Thus \( N = \frac{T'(\pi)}{|T'(\pi)|} = \frac{1}{4} < 0, 0, -4 > = < 0, 0, -1 > \) and \[
    B = T \times N = \frac{1}{\sqrt{5}} \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \frac{1}{\sqrt{5}} < -1, 2, 0 >
\]