Problem 1. [20pts] Evaluate each of the following limits:

1. \[ \lim_{x \to +\infty} \frac{\ln (\ln x)}{\ln x} = \frac{\infty}{\infty} \] indeterminate! We will use Hospital’s rule then.

\[ \lim_{x \to +\infty} \frac{\ln (\ln x)}{\ln x} = H \lim_{x \to +\infty} \frac{[\ln (\ln x)]'}{[\ln x]'} = \lim_{x \to +\infty} \frac{\frac{(\ln x)'}{\ln x}}{\frac{1}{x}} = \lim_{x \to +\infty} \frac{1}{x} \ln x = \lim_{x \to +\infty} \frac{1}{\ln x} = 0.1 \]

2. \[ \lim_{x \to 0^+} \frac{(\sin x)^{\tan x}}{x} = 0^0, \text{ indeterminate! We will use the \textquotedblleft ln\textquotedblright{} transformation and Hospital’s rule.} \]

Put \( y = (\sin x)^{\tan x} \). Then \( y = \tan x \ln (\sin x) \). We have:

\[ \lim_{x \to 0^+} y = \lim_{x \to 0^+} \tan x \ln (\sin x) = \lim_{x \to 0^+} \frac{\ln (\sin x)}{\frac{1}{\tan x}} = H \lim_{x \to 0^+} \frac{[\ln (\sin x)]'}{[\frac{1}{\tan x}]'} = \lim_{x \to 0^+} \frac{\cos x}{\sin x} \cdot \frac{\sin x}{\cos^2 x} / \tan^2 x = \lim_{x \to 0^+} \frac{\cos x}{\sin x} = 0. \]

Therefore, \( \lim_{x \to 0^+} y = e^{0} = 1. \]

Problem 2. [30pts] The figure below shows the graph of the derivative \( f' \) of \( f \): (it is the graph of \( f' \)), defined only on the interval \([0, 4]\). (Click on the link “its graphs” to see the graph of \( f' \)).

1. On what intervals is \( f \) increasing on \([0, 4]\)? why?

On \([2, 3]\) because the derivative \( f' \) is positive.

2. On what intervals is \( f \) decreasing on \([0, 4]\)? Why?

On \([0, 2]\) and \([3, 4]\) because the derivative \( f' \) is negative.

3. At what values of \( x \) does \( f \) have local maximum on the interval \([0, 4]\)? Why?

\(-\) At the critical number 3, because \( f'(3) = 0 \) and \( f' \) changes sign from positive to negative at 3.

4. At what values of \( x \) does \( f \) have local minimum on the interval \([0, 4]\)? Why?

\(-\) At the critical number 2, because \( f'(2) = 0 \) and \( f' \) changes sign from negative to positive at 2.

5. Find, approximately, the point where \( f \) increases most rapidly on the interval \([0, 4]\)? Justify your answer!

The derivative \( f' \) gives the rate of change of \( f \). Hence, \( f \) increases most rapidly when the rate of change \( f' \) is the largest and positive. So we need to find the maximum of \( f' \). On the interval \([0, 4]\), \( f' \) has its maximum value at \( x = 2.5 \).

6. On what intervals is \( f \) concave up on \([0, 4]\)? why?

Using the derivative \( f'' \), \( f \) is concave up when \( f' \) is increasing (\( f'' \) positive). So, \( f \) is concave up on the intervals : \([1, 2.5]\) and \([3.65, 4]\). (Any value close to 3.65 is OK)

7. On what intervals is the second derivative \( f'' \) is negative on \([0, 4]\)? why?

\( f' \) is decreasing \( \iff \) \( f'' \) is negative. So, \( f'' \) is negative on the intervals : \([0, 1]\) and \([2.5, 3.65]\)

8. What are the inflections points of \( f \) on the interval \([0, 4]\)? Why?

They are where \( f \) changes concavity. So, they are the points where \( f' \) changes from increasing to decreasing and vice-versa: \( \{x = 1, x = 2.5\}, \text{ and } x = 3.65 \} \)
9. Find, approximately, the point where $f''$ is the greatest on the interval $[0, 4]$? Justify your answer!

Since the derivative is also the slope, we are looking for $x$ where $f'$ has the largest positive slope (i.e., $f'$ increases most rapidly) on the interval $[0, 4]$. From the graph of $f'$, it occurs (approximately) at $x = 4$.

Problem 3. [20pts.] Suppose you run a small independent furniture business. You sign a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be $80 per chair up to 300 chairs, and above 300, for every additional chair over 300 ordered, the price will be reduced by $0.20 per chair on the whole order. What is the number of chairs that will maximize the revenue of your company if the customer choose to order more than 300 chairs?

Let $x$ = the number of extra chairs above 300.
Let $R$ = the revenue = quantity × price. Then:

\[
\begin{align*}
\text{price} & = 80 - x \cdot (0.20) = 80 - 0.2x \\
\text{quantity} & = \text{number of chairs} = 300 + x.
\end{align*}
\]

Hence $R(x) = (300 + x)(80 - 0.2x) = 24,000 + 20x - 0.2x^2$ to maximize! Let's find the critical numbers:

\[
R'(x) = 20 - 0.4x = 0 \Rightarrow x = \frac{20}{0.4} = 50 \text{ is the only critical number.}
\]

Moreover, $R''(x) = -0.4 \Rightarrow R''(50) = -0.4 < 0$. By the second derivative test, $R$ has a local maximum at $x = 50$, which is an absolute maximum since it is the only critical number.

The number of chairs that will maximize the revenue is then $300 + 50 = 350$.

Problem 4. [20pts.] A ball is dropped from a tower at the time $t = 0$. When it strikes the ground it is travelling downward at a speed of 60 feet per second.

1. Give the formula of the velocity $V(t)$ as a function of time.

We have only the acceleration due to gravitational attraction. Therefore $a(t) = g = -32 \text{ft/s}^2$.

Hence, $V(t) = -32t + C$. Since $V = 0$ at $t = 0$, then by substitution: $0 = -32(0) + C \Rightarrow 0 = C$.

Thus $V(t) = -32t$.

2. How long does it take the ball to reach the ground?

When it reaches the ground, its velocity is: $V = -60 \text{ft/s}$ (negative because the downward direction). Therefore, we solve for $t$, $V(t) = -60$. Thus, $-32t = -60 \Rightarrow t = \frac{-60}{-32} = 1.875$ seconds.

3. How tall is the tower?

Since $V(t) = -32t$, then, by taking the antiderivative, the position is given by $S(t) = -32 \frac{t^2}{2} + C = -16t^2 + C$. When the ball strikes the ground its position is 0 and the corresponding time is $t = 1.875$ seconds. Therefore, to find the constant $C$, we solve $S(1.875) = 0$.

$S(1.875) = 0 \Rightarrow -16 \cdot 1.875^2 + C = 0 \Rightarrow C = 16 \cdot 1.875^2 = 56.25$

Hence, $S(t) = -16t^2 + 56.25$. At $t = 0$, the position of the ball is $S(0) = 56.25$, therefore, the tower is 56.25 feet tall.

Problem 5. [15pts.] Consider a function $f$, with domain $(-\infty, 6)$ and $(6, +\infty)$, that satisfies the following conditions:

1. Use the information above to complete this table of variation of the function $f$, like we did in the class, showing for example where $f$ is increasing, decreasing, concave up or down, domain, asymptotes, etc.

2. Sketch the possible graph of $f$. 