Problem 1. Find the derivatives for each of the following functions using the differentiation rules. Do not simplify! but you have to show your work with all the necessary steps to receive credit.

1. \( f(x) = \frac{x^7 + 9}{\cos(3x)} \)

\[
f'(x) = \frac{(x^7 + 9)'(\cos(3x)) - (x^7 + 9)(\cos(3x))'}{\cos(3x))^2}
\]
\[
= \frac{7x^6 \cdot \cos(3x) + (x^7 + 9) \cdot 3 \sin(3x)}{\cos(3x))^2}
\]
quotient rule

2. \( f(x) = 3^{x^\pi} \)

Since the base is a constant we can use the rule: \((a^W)' = \ln a \cdot a^W \cdot W'\) with \(a = 3\) and \(W = x^\pi\)

\[
f'(x) = \ln 3 \cdot 3^{x^\pi} (x^\pi)'
\]
\[
= \ln 3 \cdot 3^{x^\pi} (\pi x^{(\pi - 1)})
\]
using the power rule for \((x^\pi)'\)

3. \( f(x) = \ln[\ln(ln x)] \)

Using the chain rule for the “\(\ln\)” function: \((\ln W)' = \frac{W'}{W}\)

\[
f'(x) = \frac{[\ln(\ln x)]'}{[\ln(\ln x)]}
\]
\[
= \frac{[\ln x]'}{\ln x}
\]
with \(W = \ln(\ln x)\)

\[
= \frac{x}{\ln x} \cdot \ln(\ln x)
\]
using \((\ln x)' = \frac{1}{x}\)

4. \( f(x) = \sqrt{x^2 + 1} \)

Since the base and the exponent are both functions of \(x\) (not a constant), we have to use the technique of the logarithmic differentiation. We start by applying the logarithm to both sides:

\[
\ln f(x) = \ln[\sqrt{x^2 + 1}]
\]
logarithm in both sides

\[
\ln f(x) = \sqrt{x^2 + 1} \ln x
\]
logarithm property

\[
[\ln f(x)]' = [\sqrt{x^2 + 1} \ln x]' \quad \text{derivative in both sides}
\]

\[
\frac{f'(x)}{f(x)} = \frac{[\sqrt{x^2 + 1}]' \ln x + \sqrt{x^2 + 1} [\ln x]'}{\sqrt{x^2 + 1} \ln x}
\]
logarithm and product derivative rules

\[
\frac{f'(x)}{f(x)} = \frac{2x}{2 \sqrt{x^2 + 1} \ln x + \sqrt{x^2 + 1} \frac{1}{x}}
\]
chain rule and basic derivatives

Hence,

\[
f'(x) = f(x) \cdot \left[ \frac{x \ln x}{\sqrt{x^2 + 1} + \sqrt{x^2 + 1} \frac{1}{x}} \right]
\]
Problem 2. Find the derivative $y'$ (or $\frac{dy}{dx}$) at the point $(x, y) = (1, 1)$ if: $x^3 + xy - 2y^3 = \sin(x - y)$

\[
\begin{align*}
[x^3 + xy - 2y^3]' &= [\sin(x - y)]' \\
3x^2 + (x'y + xy') - 2(3y^2y') &= (x - y)' \cos(x - y) \\
3x^2 + y + xy' - 6y^2y' &= (1 - y') \cos(x - y)
\end{align*}
\]

So for $x=1$ and $y=1$ we have:

\[31^2 + 1 + 1y' - 61^2y' = (1 - y') \cos(1 - 1)\]
\[4 - 5y' = 1 - y'\]
\[-4y' = -3\]
\[y' = \frac{3}{4}\]

Problem 3. The following figure shows the graphs of a function $f$, its derivative $f'$, and its second derivative $f''$. Identify each graph, and explain briefly your choices.

Note that $a' = c$, since $c = 0$ when $a$ has a horizontal slope, $c > 0$ when $a$ increasing and $c < 0$ when $a$ decreasing. In the other hand, $c' = b$ since $b = 0$ when $c$ has a horizontal slope, $b > 0$ when $c$ increasing and $b < 0$ when $c$ decreasing. Therefore, $f = a$, $f' = c$ and $f'' = b$.

Problem 4. The position of a particle, moving along a straight line, is given by $s = f(t) = t^3 - 6t^2 + 3$ for $0 \leq t \leq 5$, where $t$ is measured in seconds and $s$ in feet.

1. What is the velocity at time $t$?
   \[v(t) = s' = f'(t) = 3t^2 - 12t\]

2. What is the acceleration at time $t$?
   \[a(t) = v'(t) = 6t - 12\]

3. When is the particle moving forward (in the positive direction)?
   We have to find when the velocity is positive for $0 \leq t \leq 5$. So we need first to solve $v(t) = 0$:
   \[v(t) = 3t^2 - 12t = 3t(t - 4) = 0 \quad \implies t = 0 \text{ or } t = 4.\]
   By checking $v(t)$ at some values of $t$ for $0 < t < 4$ and for $4 < t \leq 5$ (for example $t = 1$ and $t = 5$), we find that $v(t) < 0$ on $[0, 4]$ and $v(t) > 0$ on $[4, 5]$.

   \[\text{the particle moving forward on } [0, 4]\]

4. Find the total distance travelled (not the displacement!) during the first 5 seconds.
   Distance travelled between $t = 0$ and $t = 4$: $|s(4) - s(0)| = |-29 - 3| = |-32| = 32$
   Distance travelled between $t = 4$ and $t = 5$: $|s(5) - s(4)| = |-22 - (-29)| = |-22 + 29| = 7$
   So, the total distance during the first 5 seconds is $32 + 7 = 39$

5. Find the intervals where the particle is speeding up.
   The particle is speeding up when $v(t)$ and $a(t)$ have the same sign.
   From question 3: We know the sign of $v(t)$: negative on $[0, 4]$ and positive on $[4, 5]$.
   Since $a(t) = 6t - 12 = 0$ if $t = \frac{12}{6} = 2$, we have: $a(t)$ is negative on $[0, 2]$ and positive on $[2, 5]$.
   It follows that $v(t)$ and $a(t)$ have the same sign on $[0, 2]$ and $[4, 5]$.
   Therefore, the particle is speeding up on $[0, 2]$ and $[4, 5]$. 
Problem 5. The quantity $q$ of certain Athletic Shoes which are sold depends on the selling price $p$ in dollars: that is, $q = f(p)$.

1. Give the meaning (in practical terms) of $f(150) = 14000$.
   When the selling price of the Athletic Shoes is $150, the quantity sold is 14 000.

2. Give the meaning (in practical terms) of $f'(150) = -100$.
   When the selling price of the Athletic Shoes is $150, the quantity sold will decrease by 100 shoes for each extra $1 increase of the selling price.

3. Give the best possible estimate of $f(155)$.
   method 1: Using the linear approximation.
   the equation of the tangent line at $x = 150$, that is to say the linearization, is
   \[ L(x) = f'(150)(x - 150) + f(150) = -100(x - 150) + 14000 \]
   Therefore, $f(155) \approx L(155) = -100(155 - 150) + 14000 = -500 + 14000 = 13500$
   method 2: Using the meaning of the derivative form question 2.
   For 155 we have $5$ more that in 150. It follows, that the quantity sold will decrease by nearly 5 times 100, in other term, by 500 shoes. Therefore, $f(155) \approx 14000 - 500 = 13500$

4. Extra question: Give the units and meaning (in practical terms) of $[f^{-1}]'(10,000) = -0.05$.
   It helps to know the units of $[f^{-1}]'$: $/$ shoes. $[f^{-1}]'(10,000) = -0.05$ means that when 10,000 shoes are sold, the selling price will decrease by $0.05 = 5$ cents for extra shoes sold.