3 Projectile Motion

Introduction

Important: Complete and submit the Lab 3 problem set on WebAssign before you leave lab today. Your instructor and lab assistant will be happy to help.

In Lab 2, you tested a model for position and velocity as functions of time for objects moving in one dimension with constant acceleration. You will now have an opportunity to test the same model in two dimensions through the study of the motion of a small spherical projectile fired from a spring-loaded launcher. You will be able to measure initial velocities (both magnitude and direction) as well as vertical and horizontal displacements and compare your measurements with model predictions. Once you have had a chance to study the system, you will be challenged to land a projectile on a target placed by your instructor.

\[
\begin{align*}
    v_{2x} &= v_{1x} + a_x \Delta t \\
    v_{2y} &= v_{1y} + a_y \Delta t
\end{align*}
\]

Figure 1: The trajectory of a projectile launched at \( \theta_1 > 0^\circ \).

Our model for the velocity of an object moving in two dimensions under constant acceleration is

\[
\begin{align*}
    v_{2x} &= v_{1x} + a_x \Delta t \\
    v_{2y} &= v_{1y} + a_y \Delta t
\end{align*}
\]
The horizontal and vertical components of the displacement of the object as functions of time are given by

\[
\Delta x = v_{1x} \Delta t + \frac{1}{2} a_x \Delta t^2
\]

\[
\Delta y = v_{1y} \Delta t + \frac{1}{2} a_y \Delta t^2
\]

where \( \Delta x = x_2 - x_1 \) and \( \Delta y = y_2 - y_1 \).

The trajectory of a projectile in “free fall” under the influence of gravity is illustrated in Fig. 1. In this case, we have \( y_2 = 0 \), \( v_{1x} = v_1 \cos \theta_1 \), \( v_{1y} = v_1 \sin \theta_1 \), \( a_x = 0 \), and \( a_y = -g \), giving

\[
v_{2x} = v_1 \cos \theta_1
\]

\[
v_{2y} = v_1 \sin \theta_1 - g \Delta t
\]

and

\[
\Delta x = (v_1 \cos \theta_1) \Delta t
\]

\[
-y_1 = (v_1 \sin \theta_1) \Delta t - \frac{1}{2} g \Delta t^2
\]

where \( g \) is the local acceleration due to gravity.

You will show for homework, by combining Eq.’s 7 and 8, that the horizontal range \( R = \Delta x \) of a projectile launched with initial speed \( v_1 \), at an angle \( \theta_1 \) with respect to horizontal, from a height \( y_1 \) and landing at height \( y_2 = 0 \) is given by

\[
R(\theta_1) = v_1 \cos \theta_1 \left( \frac{v_1 \sin \theta_1}{g} + \sqrt{\left( \frac{v_1 \sin \theta_1}{g} \right)^2 + \frac{2y_1}{g}} \right)
\]

This is the key equation for finding a launch angle that will give the projectile a particular horizontal range. Unfortunately, Eq. 9 cannot be solved for \( \theta_1 \) algebraically. You will use a spreadsheet to solve it numerically instead.

**Experiments**

You have been supplied with a projectile launcher, a plastic ball (your projectile), a meter stick, and a sheet of carbon paper. According to the manufacturer of the launcher, the launch position \((x_1, y_1)\), the position at which the ball loses contact with the spring and enters a state of free-fall, does not vary with launch angle. However, the initial speed imparted to projectiles may vary as much as 8% over the available range of launch angles. Use the medium range setting of the launcher for all measurements.

**Preliminary Launches**

1. Measure \( y_1 \), the height above the table at which the projectile is launched. Be careful here. You want the magnitude of the actual vertical displacement of the ball.
2. Launch the projectile at $\theta_1 = 0^\circ$ with respect to the horizontal. (See Fig. 2.) It should land on the table before bouncing away. If it doesn’t, mount the launcher further from the end of the table and try again.

3. Now you know approximately where the ball hits the table for a horizontal launch. Tape a blank sheet of paper over this “landing zone,” and lay a sheet of carbon paper over it. If you have placed the paper properly the projectile will touch down on the carbon, leaving a mark on the paper. You can then use a meter stick to determine the horizontal range $\Delta x$ of the projectile.

4. Launch the projectile several times to produce a “hit pattern” which gives you a sense of the reproducibility of the horizontal range.

5. Then, set the launch angle at $\theta_1 = 80^\circ$, and measure the maximum range for several launches. The point of these two initial experiments is to give you a chance to test both the launcher and the theoretical model so that you can make an informed prediction for the “Target Practice” experiment. **Skip to the Analysis section before proceeding.**

**Target Practice**

Now that you have studied the launcher and your theoretical model, see if you can use what you have learned to hit a target. Your instructor will tape a target to the lab table. Landing the projectile within the boundaries of the target on the first try is worth extra credit. Have your instructor witness your attempt. You may use only the meter stick and a spreadsheet to choose your launch angle. Trial shots (beyond your preliminary launches at $\theta_1 = 0^\circ$ and $80^\circ$) disqualify you for extra credit and invite a grade penalty.

**Analysis**

**Initial Speed from a Horizontal Launch**

The trajectory of a projectile launched at $\theta_1 = 0^\circ$ is shown in Fig. 2. You can determine $v_1$ from your measurements of the horizontal and vertical displacements of the projectile $\Delta y$ and $\Delta x$. The time it takes the projectile to drop to the table is found using Equation 8. The projectile was launched horizontally, so $v_{1y} = 0$. Hence, time at which $y = 0$ (i.e. when it hits the table) is given by

$$\Delta t_{\text{max}} = \sqrt{\frac{2y_1}{g}}. \quad (10)$$

The initial velocity is then given by plugging $\Delta t_{\text{max}}$ into Equation 7,

$$v_1 = v_{1x} = \frac{\Delta x}{\Delta t_{\text{max}}}. \quad (11)$$
Launch at an Angle

In order to determine the launch angle you will use to try to hit the target, program Eq. 9 in a spreadsheet so that you can adjust the parameters easily. Compare the predictions of this theoretical model with your measured horizontal ranges with \( \theta_1 = 80^\circ \). How well do they compare? Is there any pattern in deviations of the model from experiment? Do you need to correct the model in order to hit a target?

If you aren’t familiar with spreadsheets or need a refresher, here is one way to program Eq. 9. Leave row A for column labels. Put the initial height in cell B1, the initial speed in cell B2, the launch angle in cell B3, and \( g \) in cell B4. Place \( \frac{v_1 \sin \theta_1}{g} \) in cell B5, which in spreadsheet syntax looks like

\[
=B2*\sin(\text{radians}(B3))/B4
\]

Finally, place the range in cell B6,

\[
=B2*\cos(\text{radians}(B3))*(B5 + \text{sqrt}(B5^2 + 2*B1/B4))
\]

The spreadsheet trig functions assume angles are expressed in radians. I use the \text{radians()} function above to convert to radians so that the launch angle in cell B3 can be entered in degrees. Organizing the calculation in one row facilitates producing several rows with different parameters \( (y_1, v_1, \theta_1, g, \text{e.g.}) \) for comparison.

Uncertainty

Your distance and angle measurements have inherent uncertainties. Estimate and record these. Use the hit patterns to quantify the uncertainty in horizontal range.

Taking Your Shot

Before you attempt to hit the target, go over your calculations with your instructor/TA and be sure to have your instructor/TA present to witness your shot.
Questions

As you formulate answers to the following questions, apply the concepts related to systematic uncertainties and uncertainties due to random variations (Appendix A) that you have been learning in lab this semester.

1. Did your launches at $\theta_1 = 80^\circ$, lead you conclude that you needed to make some kind of correction to the model in order to hit a target? Explain. If you made a correction, describe it.

2. Do the uncertainties in your meter stick and plumb bob measurements explain the random scatter in the observed hit patterns? Explain. If not, what else might be going on?

3. Based on your work, what minimum target size can you reliably hit with your launcher?

4. If you hit the target, was there any luck involved? If you missed the target, why do you think you missed? Did you make a mistake or were you a victim of random variation?