Mapping Cone Sequences and a Generalized Notion of Cone Length

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Abstract

We introduce a weighted length between spaces. This is accomplished by using the numerical invariants of cone length and killing length as a framework and by considering other topological invariants to determine the complexity of spaces. This leads us to the definition of a weighted length between spaces. We estimate the weighted length amongst certain maps and spaces for pushouts, pullbacks, and fibrations. Examples of specific weights are given to show that hypotheses in theorems are necessary.

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1 Introduction

Let $\mathcal{A}$ be a collection of spaces with $* \in \mathcal{A}$ and the property that if $A \in \mathcal{A}$ and A has the same homotopy type as B, then $B \in \mathcal{A}$. The $\mathcal{A}$-*cone length* of a space X, denoted $\mathrm{cl}_{\mathcal{A}}(X)$, is the minimum number of cones needed to build X from $*$, where each cone is on a space in $\mathcal{A}$. This topological invariant was introduced by Clapp and Puppe [4] as a generalization of the cone-length of X([5] Chapter 3.5). Closely related to the notion of $\mathcal{A}$-cone length is the $\mathcal{A}$-*killing length* of a space X, denoted $\mathrm{kl}_{\mathcal{A}}(X)$ first studied in [3]. This is the minimum number of cones needed to build $*$ from X where each cone is on a space in $\mathcal{A}$.

In this paper, we use the cone length and killing length to introduce a further generalization, the *weighted length* between spaces, denoted $\ell^{\omega}(X, Y)$. We define this in Section 2. A *weight function* ω assigns a nonnegative real number, called a *weight*, to each space attached; that is, each space A we attach will have a certain number $\omega(A)$ associated to it. A length between maps is defined in order to induce a *weighted length* between X and Y which will be the infimum of the sum of all spaces attached over all possible decompositions (maps) of X into Y. The *weight function* ω should take into account the complexity of the spaces that are attached. We propose a few properties that a weight function should satisfy in Definition 1. These axioms are minimally restrictive, allowing a plethora of possible weight functions. In Section 3, we prove an equivalent definition (Definition 6) of the length between maps. Section 4 uses results of Arkowitz, Stanley, and Strom [2] to deduce relations between weighted length of spaces in homotopy pushouts, pullbacks, and fibrations. For example, we prove

Corollary 21 *Let ω be a bounded weight function with the property that $\omega(X * Y) \leq \omega(X)$ for all spaces X, Y. If $F \to E \to B$ is a fibration and $\mathrm{cl}^{\omega}(B)$ has step size n, then $\mathrm{cl}^{\omega}(E) \leq \mathrm{cl}^{\omega}(F)(n+1) + \mathrm{cl}^{\omega}(B)$.*

This is the natural analogue of Varadarajan's Theorem [11] which says that $\mathrm{cl}(E)+1 \leq (\mathrm{cl}(F)+1)(\mathrm{cl}(B)+1)$. After developing the general theory of weighted length, we will briefly consider three examples of weight functions, all of which provide interesting examples of why certain hypotheses in the general theory are necessary.

Of particular interest is computing the cone length and killing length of a space with respect to a weight. We will see in Corollary 15 (ii) that $\mathrm{kl}^{\omega}(X) \leq \mathrm{cl}^{\omega}(X)$ whenever ω is bounded and satisfies $\omega(\Sigma X) \leq \omega(X)$ for all $X \in \mathcal{A}$. If ω does not satisfy this property, then the inequality may not hold, as shown in Remark 31.

In Section 5.1, we compute the weighted cone length of X where X is any simply connected CW complex. In fact, this turns out to be equal to the weighted killing length.

Theorem 24 *Let X be finite dimensional simply connected CW complex and $\omega = \omega_D$ the number of nontrivial (reduced) homology groups of X. Then $\mathrm{kl}^{\omega}(X) = \mathrm{cl}^{\omega}(X) = \omega(X)$.*

In Section 5.3 we see an example of a weight function ω for which there exists a space X such that $\mathrm{kl}^{\omega}(X) < \omega(X)$. This is due to a result of Cuvilliez and Félix found in [6].

2 Weight Functions

In this section we establish the basic definitions and concepts that will be used in the body of the paper. We use $\equiv$ to denote homotopy equivalence between spaces. We use $*$ to denote a contractible space. We will work with the class of based CW complexes whose base point is the vertex, and morphisms continuous base-point-preserving maps, *i.e.* if $f\colon X \to Y$, then $f(*_X) = *_Y$. This collection we denote by $\mathcal{C}$. Some of the theory will be applicable to all of $\mathcal{C}$. Section 5.1 will be applicable to those members of $\mathcal{C}$ which are finite dimensional and simply connected.

One of the main tools we use in this paper is a mapping cone sequence, $A \xrightarrow{f} X \longrightarrow C_f$. We say that C_f is X union a cone on A. Less formally, we say that we attach A to X where it is understood that we mean "a cone on A."

We now define a weight function and use this definition, motivated by the framework of $\mathcal{A}$-cone and killing length, to define a weighted length between spaces in Definition 3.

Definition 1 *Let $\mathcal{A}$ be a collection of spaces containing $*$ which is closed taking suspensions and wedges such that if $A \in \mathcal{A}$ and $A \equiv B$, then $B \in \mathcal{A}$. A* weight function $\omega\colon \mathcal{A} \to \mathbb{R}^{\geq 0}$ *is any function such that*

(a) $\omega(*) = 0$

(b) $\omega(A_1 \vee A_2) \leq \omega(A_1) + \omega(A_2)$ *for all spaces* A_1, A_2

(c) $\omega(X) = \omega(Y)$ *whenever* $X \equiv Y$.

In addition, if ω satisfies $\omega(\Sigma X) \leq \omega(X)$ for all spaces X, we say that ω is a Σ-weight function. *If $\omega(X) \leq C$ for some constant C, then we say that ω is a* bounded *weight function.*

For example, define $\omega_D(X) = \mathcal{H}(X)$, the number of dimensions for which X has nontrivial (reduced) homology groups whenever X is 1-connected and of finite type. It is easy to see that $\omega_D(X)$ is a Σ-weight function which is not bounded in general. See Section 5.1.

Definition 2 *Let $f\colon X \to Y$ be a map, ω a weight function, and $\mathcal{A} \subseteq \mathcal{C}$ a collection of spaces. If f is a homotopy equivalence, set $\ell^\omega(f) = 0$. Otherwise, define an* ω-decomposition of f of step-size $m < \infty$ *as a homotopy commutative diagram D*

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{m-1} \arrow[d] & \\
X_0 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] & \dots \arrow[r] & X_{m-1} \arrow[r] & X_m \arrow[d, equal] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

where each $A_i \longrightarrow X_i \longrightarrow X_{i+1}$ *is a mapping cone sequence with $A_i \in \mathcal{A}$. Set $\ell^\omega_D(f) = \sum_{i=0}^{m-1} \omega(A_i)$. The* ω-length of f *is the number $\widetilde{\ell^\omega}(f) = \inf_D\{\ell^\omega_D(f)\}$ where the* inf *is taken over all such decompositions D of finite step-size. If no such diagram D exists, we say that $\widetilde{\ell^\omega}(f) = \infty$. Any such diagram above will also be referred to as an* ω-decomposition of X into Y *or a* decomposition of X into Y *when ω is clear.*

We will see that the standard notions of cone length and killing length are recovered when $\omega(X) = 1$.

Because ω may take on noninteger values, there may be an infinite sequence of ω-decompositions of f which have values that converge to some number not realized by any ω-decomposition (see Example 12 and [[10] Example 2.23]). Hence, for $\widetilde{\ell^\omega}(f) < \infty$, let $\epsilon > 0$ be given. Then there is a decomposition D_ϵ such that $\ell^\omega_{D_\epsilon}(f) \leq \widetilde{\ell^\omega}(f) + \epsilon$. The diagram D_ϵ is called an *ϵ-approximation of f*.

Definition 3 *Let X and Y be spaces and ω a weight function. Define the* weighted length from X to Y *or* ω-length from X to Y *by $\ell^\omega(X, Y) = \inf_f\{\widetilde{\ell^\omega}(f)\}$ where f is any map from X to Y.*

Let $\ell^\omega(X,Y) < \infty$. Then given $\epsilon > 0$ there exists a map $f_\epsilon \colon X \to Y$ and a decomposition D_ϵ such that $\ell_{D_\epsilon}(f_\epsilon) \le \ell^\omega(X,Y) + \epsilon$. The diagram D_ϵ is called an *ϵ-decomposition of X into Y*. We sometimes write $D_\epsilon = (X,Y;A_i)$ or $D_\epsilon = (X,Y;A_i,j_i,i=0\dots n-1)$ as shorthand for the above diagram. We define $\mathrm{kl}^\omega(X) = \ell^\omega(X,*)$ and $\mathrm{cl}^\omega(X) = \ell^\omega(*,X)$.

Remark 4 Note that it is not the case in general that $\ell^\omega(X,Y) = \ell^\omega(Y,X)$. In particular, it is seen that if $\mathcal{A}$ is a collection of spaces and $\omega(X) = 1$ for all $X \in \mathcal{A}$ that we recover the $\mathcal{A}$-cone and killing length of the space X, $\mathrm{kl}^\omega(X) = \mathrm{kl}(X)$ and $\mathrm{cl}^\omega(X) = \mathrm{cl}(X)$, as studied in by Arkowitz, Stanley, and Strom in [1], [2], and Cuvilliez and Félix in [6] amongst other places. As noted above, we will usually work with $\mathcal{A} = \mathcal{C}$.

3 An Equivalent Definition

We give an alternate characterization of $\widetilde{l^\omega}(f)$.

Definition 5 *We say that (i,j) is a* homotopy equivalence *from f to f' (and (r,s) is a homotopy equivalence from f' to f) if there is a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[r, "r"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"] & Y' \arrow[r, "s"] & Y \end{tikzcd}$$

where $ri \simeq \mathrm{id}, sj \simeq \mathrm{id}, ir \simeq \mathrm{id}$, and $js \simeq \mathrm{id}$ and write $f \equiv f'$.

The following Definition and proof of Proposition 7 are extensions of the ideas in [2].

Definition 6 *Let ω be a bounded weight function and L^ω be a function such that for every $f \colon X \to Y$, $L^\omega(f) \in [0,\infty]$ satisfies*

- *(a) $L^\omega(f) = 0$ whenever f is a homotopy equivalence.*
- *(b) If $A \longrightarrow X \xrightarrow{f} Y$ is a mapping cone sequence, then $L^\omega(f) \le \omega(A)$.*
- *(c) $L^\omega(fg) \le L^\omega(f) + L^\omega(g)$*
- *(d) If $f \equiv g$, then $L^\omega(f) = L^\omega(g)$.*

Define $\mathcal{L}^\omega(f) = \sup\{L^\omega(f) : L^\omega$ satisfies the above properties $\}$. Let $\epsilon > 0$ and f be given. If $\mathcal{L}^\omega(f)$ is not infinite, then there is an L^ω_ϵ such that $\mathcal{L}^\omega(f) \le L^\omega_\epsilon(f) + \epsilon$. Note that L^ω_ϵ depends on both f and ϵ.

Proposition 7 *Let $f \colon X \to Y$ be any map and ω a bounded weight function. Then $\widetilde{\ell^\omega}(f) = \mathcal{L}^\omega(f)$.*

Proof Recall that $\widetilde{\ell^\omega}(f) = \inf_D\{\ell^\omega_D(f)\}$ where the inf is taken over all decompositions D of f. We first show that if $\mathcal{L}^\omega(f)$ is finite, then $\mathcal{L}^\omega(f) \le \widetilde{\ell^\omega}(f)$. Let $\epsilon > 0$ and $L^\omega_{\epsilon/2}$ a function such that $\mathcal{L}^\omega(f) \le L^\omega_{\epsilon/2}(f) + \frac{\epsilon}{2}$. Then there is an $\frac{\epsilon}{2}$-decomposition $D_{\epsilon/2}$ of f such that $\ell^\omega_{D_{\frac{\epsilon}{2}}}(f) \le \widetilde{\ell^\omega}(f) + \frac{\epsilon}{2}$ (where it is understood that $\widetilde{\ell^\omega}(f)$ could be infinite).

Write

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\
X_0 \arrow[r, "j_0"] \arrow[d, equal] & X_1 \arrow[r, "j_1"] & \dots \arrow[r] & X_{n-1} \arrow[r, "j_{n-1}"] & X_n \arrow[d, equal] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

for $D_{\frac{\epsilon}{2}}$. Observe that the diagram

$$\begin{tikzcd}
X_0 \arrow[r, equal] \arrow[d, "j_{n-1}\cdots j_0"] & X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, "j_{n-1}\cdots j_0"] \\
X_n \arrow[r, equal] & Y \arrow[r, equal] & X_n
\end{tikzcd}$$

shows that $f \equiv j_{n-1}\dots j_0$. Hence

$$\begin{aligned}
\mathcal{L}^\omega(f) \leq L^\omega_{\epsilon/2}(f) + \frac{\epsilon}{2} &\leq L^\omega_{\epsilon/2}(j_{n-1}) + \dots + L^\omega_{\epsilon/2}(j_0) + \frac{\epsilon}{2} \\
&\leq \omega(A_{n-1}) + \dots + \omega(A_0) + \frac{\epsilon}{2} \\
&= \ell^\omega_{D_{\epsilon/2}}(f) + \frac{\epsilon}{2} \\
&\leq \widetilde{\ell^\omega}(f) + \epsilon.
\end{aligned}$$

Letting ϵ tend to 0, we see that $\mathcal{L}^\omega(f) \leq \widetilde{\ell^\omega}(f)$.

Suppose that $\mathcal{L}^\omega(f)$ is infinite. We wish to show that $\widetilde{\ell^\omega}(f) > N$ for any $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$. Since $\mathcal{L}^\omega(f)$ is infinite, there is an L^ω such that $L^\omega(f) > N$. If $\widetilde{\ell^\omega}(f)$ is finite, then there is an $\epsilon > 0$ and a decomposition D_ϵ of f such that $\ell^\omega_{D_\epsilon}(f) \leq \widetilde{\ell^\omega}(f) + \epsilon$. Using the exact same argument above, we see that if $\mathcal{L}^\omega(f)$ is infinite then so is $\widetilde{\ell^\omega}(f)$.

Next we show that $\widetilde{\ell^\omega}(f)$ satisfies Definition 6.

(i) If f is a homotopy equivalence, then $\widetilde{\ell^\omega}(f) = 0$ by definition.

(ii) If $A \longrightarrow X \xrightarrow{f} Y$ is a mapping cone sequence, then by definition $\ell^\omega_D(f) = \omega(A)$ and $\widetilde{\ell^\omega}(f) \leq \ell^\omega_D(f) = \omega(A)$ where D is the decomposition

$$\begin{tikzcd}
A \arrow[d] & \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

(iii) Let $X \xrightarrow{g} Y \xrightarrow{f} Z$ and suppose that both $\widetilde{\ell^\omega}(f), \widetilde{\ell^\omega}(g)$ are finite. Let $\epsilon > 0$. Then there is a decomposition $G_{\epsilon/2} = (X, Y; A_i, i = 0, 1, \dots n-1)$ such that $\ell^\omega_{G_{\epsilon/2}}(g) \leq \widetilde{\ell^\omega}(g) + \frac{\epsilon}{2}$. Similarly, there exists

$F_{\epsilon/2} = (Y, Z; B_i, i = 0, 1, \ldots, m-1)$ such that $\ell^{\omega}_{F_{\epsilon/2}}(f) \le \widetilde{\ell}^{\omega}(f) + \frac{\epsilon}{2}$. The following concatenation

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & B_0 \arrow[d] & & B_{m-1} \arrow[d] & \\
X_0 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] & \ldots \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] \arrow[d, equal] & \ldots \arrow[r] & X_{n+m-1} \arrow[r] & X_{n+m} \arrow[d, equal] \\
X \arrow[rrrr, "g"] & & & & Y \arrow[rrr, "f"] & & & Z
\end{tikzcd}$$

is an ω-decomposition of fg so that

$$\begin{aligned}
\widetilde{\ell}^{\omega}(fg) \le \ell^{\omega}_{D_\epsilon}(fg) &= \ell^{\omega}_{F_\epsilon}(f) + \ell^{\omega}_{G_\epsilon}(g) \\
&\le \widetilde{\ell}^{\omega}(f) + \widetilde{\ell}^{\omega}(g) + \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$, $\widetilde{\ell}^{\omega}(fg) \le \widetilde{\ell}^{\omega}(f) + \widetilde{\ell}^{\omega}(g)$.

Thus if both $\widetilde{\ell}^{\omega}(f), \widetilde{\ell}^{\omega}(g)$ are finite, then $\widetilde{\ell}^{\omega}(fg)$ is also finite so that if $\widetilde{\ell}^{\omega}(fg)$ is infinite, one of $\widetilde{\ell}^{\omega}(f), \widetilde{\ell}^{\omega}(g)$ must also be infinite, and the inequality holds in the infinite case as well.

(iv) Finally, assume $f \equiv g$. Then Lemma 3.4 of [1] shows that if

$$\begin{tikzcd}
L_0 \arrow[d] & L_1 \arrow[d] & & L_{n-1} \arrow[d] & \\
X_0 \arrow[r, "j_0"] \arrow[d, equal] & X_1 \arrow[r, "j_1"] & \ldots \arrow[r] & X_{n-1} \arrow[r, "j_{n-1}"] & X_n \arrow[d, equal] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

is an ω-decomposition of f, then one can construct spaces X'_i for $0 \le i \le n$ and maps j'_i for $0 \le i \le n-1$ that fit into an ω decomposition of g of the form

$$\begin{tikzcd}
L_0 \arrow[d] & L_1 \arrow[d] & & L_{n-1} \arrow[d] & \\
X'_0 \arrow[r, "j'_0"] \arrow[d, equal] & X'_1 \arrow[r, "j'_1"] & \ldots \arrow[r] & X'_{n-1} \arrow[r, "j'_{n-1}"] & X'_n \arrow[d, equal] \\
X \arrow[rrrr, "g"] & & & & Y
\end{tikzcd}$$

so that $\ell^{\omega}(g) \le \ell^{\omega}(f)$. The same argument shows that $\ell^{\omega}(f) \le \ell^{\omega}(g)$. Hence $\widetilde{\ell}^{\omega}(f) = \widetilde{\ell}^{\omega}(g)$ whenever $f \equiv g$.

Thus $\widetilde{\ell}^{\omega}(f)$ satisfies Definition 6 and so $\widetilde{\ell}^{\omega}(f) \le \mathcal{L}^{\omega}(f)$. We conclude that $\widetilde{\ell}^{\omega}(f) = \mathcal{L}^{\omega}(f)$. $\square$

Corollary 8 *If ω is a bounded weight function, $\ell^{\omega}(X, Y) = \inf_f\{\mathcal{L}^{\omega}(f)\}$.*

We thus have two different ways to think about the weighted length of a map. We will go back and forth between the two definitions through this paper as needed.

Corollary 9 *Let ω be a bounded weight function. For any spaces X, Y, Z, we have $\ell^\omega(X,Z) \leq \ell^\omega(X,Y) + \ell^\omega(Y,Z)$.*

Corollary 10 *For any bounded ω-function, $\ell^\omega(X,Y) \leq \mathrm{kl}^\omega(X) + \mathrm{cl}^\omega(Y)$.*

Proof This follows from the triangle inequality $\ell^\omega(X,Y) \leq \ell^\omega(X,*) + \ell^\omega(*,Y)$. $\square$

Corollary 10 gives a rudimentary upper bound for estimating the weighted length in terms of cone length and killing length.

In the case that ω is a Σ-weight function, we obtain an easy result.

Proposition 11 *Let ω be a Σ-weight function. Then $\ell^\omega(\Sigma X, \Sigma Y) \leq \ell^\omega(X,Y)$.*

Proof Let

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\
X_0 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] & \dots \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[d, equal] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

be any ϵ-approximation of X into Y so that $\sum_{i=0}^{n-1} \omega(A_i) \leq \ell^\omega(X,Y) + \epsilon$. Now $\omega(\Sigma A) \leq \omega(A)$ and $\sum_{i=0}^{n-1} \omega(\Sigma A_i) \leq \sum_{i=0}^{n-1} \omega(A_i)$. Since attaching the ΣA_i provides an ϵ-approximation of ΣX into ΣY, $\ell^\omega(\Sigma X, \Sigma Y) \leq \sum_{i=0}^{n-1} \omega(\Sigma A_i) \leq \sum_{i=0}^{n-1} \omega(A_i) \leq \ell^\omega(X,Y) + \epsilon$. In other words, we suspend the above decomposition diagram. Thus letting $\epsilon \to 0$, $\ell^\omega(\Sigma X, \Sigma Y) \leq \ell^\omega(X,Y)$. $\square$

We now give an example of a weight function ω for which there exists a space X such that $\mathrm{kl}^\omega(X)$ is not attained by any decomposition. A more naturally occuring example of a weight function ω' for which $\ell^{\omega'}(X,Y) = 0$ is not attained by any decomposition D for all spaces X, Y is given in [10] Example 2.23.

Example 12 *Let X be a space of finite type with $\cup(X) \geq 3$, where $\cup(X)$ denotes the cup length of X. Define a weight function ω as follows: For*

$S^n, n \geq 2$, define

$$\begin{aligned}
\omega(S^2) &= 3.2\\
\omega(S^3) &= 3.15\\
\omega(S^4) &= 3.142\\
\omega(S^5) &= 3.1416\\
&\vdots\\
\omega(S^n) &= 3.1415926\ldots k_{n-2}k_{n-1} + \frac{1}{10^{n-1}}\\
&\vdots
\end{aligned}$$

where k_i is the i^{th} digit in the decimal expansion of π. Define $\omega(\Sigma^j(S^i \vee X)) = 4$ for $2 \leq i$ and $0 \leq j$, and $\omega(Y) = 10$ for any space Y not defined above. Then ω is seen to be a bounded Σ-weight function. We show that $\mathrm{kl}^\omega(X) = 4 + \pi$.

Consider $\ell^\omega(X, *) = \mathrm{kl}^\omega(X)$. Clearly $X \to X \to *$ provides an upper bound for $\mathrm{kl}^\omega(X) \leq 10$. The diagram

$$\begin{tikzcd}
S^2 \vee X \arrow[d, "{(*,\mathrm{id})}"] & S^3 \arrow[d] & \\
X \arrow[r] & S^3 \arrow[r] & *
\end{tikzcd}$$

yields an estimate of $\mathrm{kl}^\omega(X) \leq 4 + 3.15$ and

$$\begin{tikzcd}
S^3 \vee X \arrow[d, "{(*,\mathrm{id})}"] & S^4 \arrow[d] & \\
X \arrow[r] & S^4 \arrow[r] & *
\end{tikzcd}$$

yields an estimate of $\mathrm{kl}^\omega(X) \leq 4 + 3.142$. In general, we have the diagram

$$\begin{tikzcd}
S^{n-1} \vee X \arrow[d, "{(*,\mathrm{id})}"] & S^n \arrow[d] & \\
X \arrow[r] & S^n \arrow[r] & *
\end{tikzcd}$$

which yields the estimate $\mathrm{kl}^\omega(X) \leq 4 + 3.14159\ldots k_{n-2}k_{n-1} + \frac{1}{10^{n-1}}$. Hence $\mathrm{kl}^\omega(X) \leq 4 + 3.14159\ldots = 4 + \pi$. Now the smallest $\mathrm{kl}^\omega(X)$ could be is $\pi + \pi = 2\pi$; that is, if there is a diagram

$$\begin{tikzcd}
S^j \arrow[d] & S^i \arrow[d] & \\
X \arrow[r] & S^i \arrow[r] & *
\end{tikzcd}$$

since the only possible 1*-step decomposition is* $X \longrightarrow X \longrightarrow *$. *Otherwise, the next smallest it could possibly be is* $4+3.14159\ldots = 4+\pi$ *which would provide the desired lower bound. Observe that the long exact cohomology sequence induced by* $S^j \to X \to S^i$ *implies that the maximum number of nontrivial cohomology groups of* X *is* 3*. Thus, the largest the cup length of* $H^*(X)$ *could be is* 2*. But* $\cup(X) \geq 3$*. Hence* $\ell^\omega(X,*) = 4+\pi$ *and* $4+\pi$ *can not be realized by any diagram* D*.*

4 Homotopy Pushouts and Pullbacks

We translate several known results into the language of length between spaces. In [2], the authors prove several results about the $\mathcal{A}$-cone length of f. If ω is a bounded Σ-weight function, then many of those results hold for $\mathcal{L}^\omega(f)$.

Theorem 13 *(cf. [2] Corollary 3.4) Let*

$$\begin{tikzcd} C \arrow[d, "c"] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\ C' & A' \arrow[l, "g'"'] \arrow[r, "f'"] & B' \end{tikzcd}$$

be a homotopy commutative diagram. Let D *be the homotopy pushout of the top row and* D' *the homotopy pushout of the bottom row and* $d\colon D \to D'$ *the induced map. If* ω *is a bounded* Σ*-weight function, then* $\mathcal{L}^\omega(d) \leq \mathcal{L}^\omega(a) + \mathcal{L}^\omega(b) + \mathcal{L}^\omega(c)$*.*

The proof of Theorem 13 may be found in [10] p.30. Several corollaries follow. We state the pertinent ones here. Others are mentioned in [10]. The proofs are all easy consequences of Theorem 13.

Corollary 14 *Let* ω *be a bounded* Σ*-weight function and* $A \to B \to C$ *be a mapping cone sequence. Then* $\mathrm{cl}^\omega(C) \leq \mathcal{L}^\omega(A \to B)$*.*

Corollary 15 *Let* ω *be a bounded* Σ*-weight function. For any map* $f\colon A \to B$

(a) $\mathcal{L}^\omega(f) \leq \mathrm{cl}^\omega(A) + \mathrm{cl}^\omega(B)$

(b) $\mathrm{kl}^\omega(A) \leq \mathrm{cl}^\omega(A)$

(c) If $f\colon A \to B$ *and* $g\colon B \to C$*, then* $\mathcal{L}^\omega(g) \leq \mathcal{L}^\omega(f) + \mathcal{L}^\omega(gf)$

(d) If $f\colon A \to B$ *and* $g\colon B \to A$ *with* $gf = \mathrm{id}$*, then* $\mathcal{L}^\omega(g) \leq \mathcal{L}^\omega(f)$

In Remark 31, we give an example of a non Σ-weight function which has the property that $\mathrm{kl}^\omega(X) > \mathrm{cl}^\omega(X)$ whenever X is a suspension.

Corollary 16 *Let* ω *be a bounded* Σ*-weight function. If* $f\colon A \to B$,

(a) $\mathcal{L}^\omega(f) \geq |\mathrm{kl}^\omega(B) - \mathrm{kl}^\omega(A)|$

(b) $\mathcal{L}^\omega(f) \geq \mathrm{cl}^\omega(B) - \mathrm{cl}^\omega(A)$

4.1 Fibrations and Length of maps

Proposition 17 *Let ω be a bounded Σ-weight function and let $i_1\colon X \to X \vee Y$ and $i_2\colon Y \to X \vee Y$ be inclusions. Then $\widetilde{\ell}^\omega(i_1) = \mathrm{cl}^\omega(Y)$ and $\widetilde{\ell}^\omega(i_2) = \mathrm{cl}^\omega(X)$.*

Proof Let $\epsilon > 0$. Then there is a diagram D_ϵ

$$\begin{tikzcd}
A_0 \arrow[d, "j_0"] & A_1 \arrow[d, "j_1"] & & A_{n-1} \arrow[d, "j_{n-1}"] & \\
Y_0 \equiv * \arrow[r] & Y_1 \arrow[r] & \dots \arrow[r] & Y_{n-1} \arrow[r] & Y_n \equiv Y
\end{tikzcd}$$

such that $\sum_{i=0}^{n-1} \omega(A_i) = \mathrm{cl}^\omega_{D_\epsilon}(Y) \leq \mathrm{cl}^\omega(Y) + \epsilon$. Let $i_k\colon Y_k \to X \vee Y_k$ for $0 \leq k \leq n-1$ and write $j_k' = i_k j_k$. From the decomposition

$$\begin{tikzcd}
A_0 \arrow[d, "j_0'"] & A_1 \arrow[d, "j_1'"] & & A_{n-1} \arrow[d, "j_{n-1}'"] & \\
X \arrow[r] \arrow[d, equal] & X \vee Y_1 \arrow[r] & \dots \arrow[r] & X \vee Y_{n-1} \arrow[r] & X \vee Y_n \arrow[d, equal] \\
X \arrow[rrrr, "i_1"] & & & & X \vee Y
\end{tikzcd}$$

we see that $\widetilde{l}^\omega(i_1) \leq \mathrm{cl}^\omega_{D_\epsilon}(Y) \leq \mathrm{cl}^\omega(Y) + \epsilon$ and therefore $\widetilde{\ell}^\omega(i_1) \leq \mathrm{cl}^\omega(Y)$. By Corollary 14 with mapping cone sequence $X \xrightarrow{i_1} X \vee Y \longrightarrow Y$, we see that $\mathrm{cl}^\omega(Y) \leq \widetilde{l}^\omega(i_1)$ which shows that $\widetilde{\ell}^\omega(i_1) = \mathrm{cl}^\omega(Y)$. The proof for i_2 is similar. $\square$

Proposition 18 *Let ω be a bounded Σ-weight function and $X \xrightarrow{*} Y$ for any spaces X and Y. Then $\mathrm{cl}^\omega(Y \vee \Sigma X) \leq \mathcal{L}^\omega(X \xrightarrow{*} Y) \leq \mathrm{cl}^\omega(Y) + \mathrm{kl}^\omega(X)$.*

Proof Since the map $X \xrightarrow{*} Y$ factors through $*$, it follows that $\mathcal{L}^\omega(X \xrightarrow{*} Y) \leq \mathrm{cl}^\omega(Y) + \mathrm{kl}^\omega(X)$. Now consider the mapping cone sequence $X \xrightarrow{*} Y \longrightarrow C_* \equiv Y \vee \Sigma X$. By Corollary 14, $\mathrm{cl}^\omega(Y \vee \Sigma X) \leq \mathcal{L}^\omega(X \xrightarrow{*} Y)$ which completes the proof. $\square$

We prove a result relating the weighted cone-lengths of the spaces in a fibration $F \to E \to B$. Although this is proved in an almost identical manner to the proof of the results in Chapter 6 of [2], we provide the details here since proofs of results concerning the cone-length with respect to a collection of spaces do not necessarily translate smoothly into proofs concerning cone-length with weighted spaces.

Recall that $A \ltimes B$ is defined as the cofiber of the map $A \xrightarrow{i_1} A \times B$ so that $C_{i_1} = A \ltimes B$. We will need the assumption that for any space X, $\omega(X * Y) \leq \omega(X)$ for all spaces Y. Although this seems like a strong condition, it is satisfied by the weight function $\omega(X) = \frac{1}{1+\mathrm{conn}(X)}$ since $\mathrm{conn}(X * Y) = \mathrm{conn}(X) + \mathrm{conn}(Y)$. This weight function is the topic of a future paper [9].

Lemma 19 *(cf. [2] Lemma 6.1) Let ω be a bounded weight function with the property that $\omega(X * Y) \leq \omega(X)$ for all spaces X, Y. If $p_2 \colon A \times B \to B$ is the projection onto B, then $\mathcal{L}^\omega(p_2) \leq \mathrm{cl}^\omega(B) + \omega(A)$.*

Proof Factor p_2 as $A \times B \xrightarrow{q} A \ltimes B \xrightarrow{p} B$. By the composition axiom, we have $\mathcal{L}^\omega(p_2) \leq \mathcal{L}^\omega(q) + \mathcal{L}^\omega(p)$. The mapping cone sequence $A \longrightarrow A \times B \xrightarrow{q} A \ltimes B$ yields $\mathcal{L}^\omega(q) \leq \omega(A)$, so it suffices to show that $\mathcal{L}^\omega(p) \leq \mathrm{cl}^\omega(B)$.

Let $\epsilon > 0$ be given and D_ϵ

$$\begin{tikzcd}
L_0 \arrow[d] & L_1 \arrow[d] & & L_{n-1} \arrow[d] & \\
B_0 \arrow[r, "j_0"] \arrow[d, equal] & B_1 \arrow[r, "j_1"] & \dots \arrow[r] & B_{n-1} \arrow[r, "j_{n-1}"] & B_n \arrow[d, equal] \\
* \arrow[rrrr] & & & & B
\end{tikzcd}$$

be an ϵ-approximation of $\mathrm{cl}^\omega(B)$ so that $\sum_{i=0}^{n-1} \omega(L_i) \leq \mathrm{cl}^\omega(B) + \epsilon$. Let $f_n \colon B_n \to B$ be a homotopy equivalence. For $0 \leq i \leq n$, consider the diagram

$$\begin{tikzcd}
A \ltimes B_i \arrow[r, "q_i"] \arrow[d, "\mathrm{id} \ltimes f_n j_{n-1} \cdots j_i"'] & B_i \arrow[r, "j_i"] \arrow[d, "r_i"] & B_{i+1} \arrow[dd, "r_{i+1}"] \\
A \ltimes B \arrow[r, "s_i"] \arrow[drr, "s_{i+1}"'] & D_i \arrow[dr, dashed, "k_i"] & \\
& & D_{i+1}
\end{tikzcd}$$

where D_i is the homotopy pushout of $\mathrm{id} \ltimes f_n j_{n-1} \dots j_i$ and q_i. Then there exists $k_i \colon D_i \to D_{i+1}$ since D_i is a pushout. For $i = n$, we have

$$\begin{tikzcd}
A \ltimes B_n \arrow[r, "q_n"] \arrow[d, "\mathrm{id} \ltimes f_n"'] & B_n \arrow[d, "r_n"'] \arrow[dr, "f_n"] & \\
A \ltimes B \arrow[r, "s_n"] \arrow[drr, "p"'] & D_n \arrow[dr, dashed, "g_n"] & \\
& & B
\end{tikzcd}$$

Since $g_n s_n = p$ and $k_i s_i = s_{i+1}$, we have $g_n k_{n-1} \dots k_0 = p$. Now r_n is a homotopy equivalence since $\mathrm{id} \ltimes f_n$ is a homotopy equivalence, and since $g_n r_n = f_n$, we see that g_n is a homotopy equivalence. We then have $\mathcal{L}^\omega(p) \leq \mathcal{L}^\omega(k_0) + \ldots + \mathcal{L}^\omega(k_{n-1})$. Form the diagram

$$\begin{tikzcd}
* \arrow[d] & A \ltimes L_i \arrow[l] \arrow[r] \arrow[d] & L_i \arrow[d] & & A * L_i \arrow[d] \\
A \ltimes B \arrow[d] & A \ltimes B_i \arrow[l] \arrow[r, "q_i"] \arrow[d] & B_i \arrow[d] \arrow[r, dotted] & {} & D_i \arrow[d, "k_i"'] \\
A \ltimes B & A \ltimes B_{i+1} \arrow[l] \arrow[r, "q_{i+1}"] & B_{i+1} & & D_{i+1}
\end{tikzcd}$$

where the columns are mapping cone sequences. The homotopy pushouts of the rows form a sequence $A * L_i \longrightarrow D_i \xrightarrow{k_i} D_{i+1}$. By a Theorem of Doeraene ([7] Lemma 1.4), this is a cofiber sequence. Thus $\mathcal{L}^\omega(p) \leq \sum_{i=0}^{n-1} \mathcal{L}^\omega(k_i) \leq \sum_{i=0}^{n-1} \omega(A * L_i) \leq \sum_{i=0}^{n-1} \omega(L_i) \leq \mathrm{cl}^\omega(B) + \epsilon$. This proves the desired result. $\square$

Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & E \arrow[d] \\
C \arrow[r] & B
\end{tikzcd}$$

be a pullback diagram and $E \to B$ a fibration with fiber F. Write $D_\epsilon = (C, B; K_i, i = 0, \ldots, n-1)$ an ϵ-approximation of $\mathcal{L}^\omega(C \to B)$. Rephrased in the language of weight functions, part of the proof of Thm. 6.2 in [2] shows that $\mathcal{L}^\omega(A \to E) \leq \sum_{i=0}^{n-1} \mathcal{L}^\omega(\ K_i \times F \xrightarrow{p_2} F\ )$. By Lemma 19, we see that $\mathcal{L}^\omega(\ K_i \times F \xrightarrow{p_2} F\ ) \leq \mathrm{cl}^\omega(F) + \omega(K_i)$. Summing over all i, we have the following.

Theorem 20 *Let*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & E \arrow[d] \\
C \arrow[r] & B
\end{tikzcd}$$

be a pullback with $F \to E \to B$ a fibration. Let ω be a bounded weight function with the property that $\omega(X * Y) \leq \omega(X)$ for all spaces X, Y. If $\mathcal{L}^\omega(C \to B)$ has step-size n, then $\mathcal{L}^\omega(A \to E) \leq n\mathrm{cl}^\omega(F) + \mathcal{L}^\omega(C \to B)$.

We end this section with the proof of our main result relating the weighted cone-length of spaces in a fibration.

Corollary 21 *Let ω be a bounded weight function with the property that $\omega(X * Y) \leq \omega(X)$ for all spaces X, Y. If $F \to E \to B$ is a fibration and $\mathrm{cl}^\omega(B)$ has step size n, then $\mathrm{cl}^\omega(E) \leq \mathrm{cl}^\omega(F)(n+1) + \mathrm{cl}^\omega(B)$.*

Proof Consider the pullback square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r] & B. \end{tikzcd}$$

By Theorem 20, $\mathcal{L}^\omega(F \to E) \leq n\mathrm{cl}^\omega(F) + \mathrm{cl}^\omega(B)$. By the composition axiom, we have

$$\begin{aligned} \mathrm{cl}^\omega(E) &\leq \mathrm{cl}^\omega(F) + \mathcal{L}^\omega(F \to E) \\ &\leq \mathrm{cl}^\omega(F) + n\mathrm{cl}^\omega(F) + \mathrm{cl}^\omega(B) \\ &= \mathrm{cl}^\omega(F)(n+1) + \mathrm{cl}^\omega(B). \end{aligned}$$

□

Remark 22 *In the case where $n = \mathrm{cl}^\omega(B)$, the inequality can be written as $\mathrm{cl}^\omega(E) + 1 \leq (\mathrm{cl}^\omega(B) + 1)(\mathrm{cl}^\omega(F) + 1)$ so that this is an appropriate analogue for weighted length.*

5 Examples

We devote this section to specific weight functions by giving examples which illustrate that certain hypotheses in the theorems above are necessary.

5.1 $\mathrm{cl}^\omega(X) = \mathrm{kl}^\omega(X) = \omega(X)$

Definition 23 *Let A be a path-connected, simply connected CW complex of finite type with reduced homology groups $H_i(A)$. Let $W(H_i(A)) = 1$ if $H_i(A) \neq 0$, 0 otherwise. Then $\omega_D(A) = \sum_{i=1}^{\infty} W(H_i(A))$ is the number of nontrivial homology groups of A. We denote this value by $\mathcal{H}(A)$. It is noted that $\omega_D(\Sigma X) = \omega_D(X)$ and that ω_D is not a bounded weight function.*

The weight function $\omega_D = \omega$ has the property that $\omega(X) = \mathrm{kl}^\omega(X) = \mathrm{cl}^\omega(X)$. We prove this now.

Theorem 24 *Let X be finite dimensional simply connected CW complex and $\omega = \omega_D$. Then $\mathrm{kl}^\omega(X) = \mathrm{cl}^\omega(X) = \omega(X)$.*

Proof
We first show that $\omega(X) \leq \mathrm{cl}^\omega(X)$. Let

$$\begin{tikzcd} A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\ * \equiv X_0 \arrow[r] & X_1 \arrow[r] & \dots \arrow[r] & X_{n-1} \arrow[r] & X_n \equiv X \end{tikzcd}$$

be any ω-decomposition of $*$ into X and let r be a dimension such that $H_r(X) \neq 0$. Let X_{k+1} be the first of the spaces $X_1, X_2, \dots, X_n$ such that $H_r(X_{k+1}) \neq 0$. Then X_{k+1} is part of a mapping cone sequence $A_k \to X_k \to X_{k+1}$ with $H_r(X_k) = 0$. Observe that in the induced long exact homology sequence

$$\ldots \longrightarrow H_r(A_k) \longrightarrow H_r(X_k) \longrightarrow H_r(X_{k+1}) \longrightarrow H_{r-1}(A_k) \longrightarrow \ldots$$

$H_{r-1}(A_k) \neq 0$. Thus for each nontrivial homology group of X, there is an attachment of a space with at least one nontrivial homology group so that $\omega(X) \leq \mathrm{cl}^\omega(X)$. Hence $\omega(X) \leq \mathrm{cl}^\omega(X)$. The proof that $\omega(X) \leq \mathrm{kl}^\omega(X)$ is similar.

Next we show that both $\mathrm{cl}^\omega(X)$ and $\mathrm{kl}^\omega(X)$ are bounded above by $\omega(X)$. It is well known that X has a Homology decomposition ([8] Chapter 10). This can be expressed as

$$\begin{tikzcd}
Z \arrow[d] & M(H_3(X),2) \arrow[d] & & M(H_n(X),n-1) \arrow[d] & \\
* \arrow[r] & X_2 \arrow[r] & \ldots \arrow[r] & X_{n-1} \arrow[r] & X_n \equiv X
\end{tikzcd}$$

where Z denotes a space with homology group $H_2(X)$ in dimension 1 and 0 elsewhere (since degree 1 Moore spaces are not unique). Since $\omega(M(H_{i+1}(X),i)) = 1$, we have that $\mathrm{cl}^\omega(X) \leq \omega(X)$.

Finally, observe that mapping cone sequence

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}"] & \\
X \arrow[r] & *
\end{tikzcd}$$

establishes that $\mathrm{kl}^\omega(X) \leq \omega(X)$.

Thus $\mathrm{kl}^\omega(X) = \mathrm{cl}^\omega(X) = \omega(X)$. □

5.2 $\mathrm{cl}^\omega(X) < \mathrm{kl}^\omega(X)$

Definition 25 *Define an ω-function ω_T on a finite dimensional CW complex by $\omega_T(X) = \frac{\mathrm{Hdim}(X)}{\mathrm{Hdim}(X)+1} = 1 - \frac{1}{\mathrm{Hdim}(X)+1}$ where $\mathrm{Hdim}(X)$ is the largest integer $n \geq 1$ such that $H_n(X) \neq 0$ and $H_i(X) = 0$ for all $i > n$. The function $\mathrm{Hdim}(X)$ is the* homological dimension *of X. If X is contractible, we say $\mathrm{Hdim}(X) = 0$.*

Since $\omega_T(\Sigma X) \nleq \omega_T(X)$, we are able to give examples of spaces X in Remark 31 such that $\mathrm{cl}^\omega(X) < \mathrm{kl}^\omega(X)$, showing that the hypothesis that ω is a Σ-weight function in Corollary 15 is necessary.

We first state a Lemma involving the relationships of the homological dimensions in a mapping cone sequence.

Lemma 26 *Let $A \to X \to Y$ be a mapping cone sequence. If $\mathrm{Hdim}(X) = \mathrm{Hdim}(Y)$, then $\mathrm{Hdim}(A) \leq \mathrm{Hdim}(X)$. Otherwise, we have that $\mathrm{Hdim}(A) = \max\{\mathrm{Hdim}(X), \mathrm{Hdim}(Y) - 1\}$.*

Lemma 27 *Let $\omega = \omega_T$. If* $\mathrm{Hdim}(X) < \mathrm{Hdim}(Y)$, *then* $1 - \frac{1}{\mathrm{Hdim}(Y)} \leq \ell^\omega(X,Y)$.

Proof Let

$$\begin{tikzcd} A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\ X \equiv X_0 \arrow[r] & X_1 \arrow[r] & \ldots \arrow[r] & X_{n-1} \arrow[r] & X_n \equiv Y \end{tikzcd}$$

be any ω-decomposition of X into Y. Proceeding from left to right, let X_{k+1} be the first space with $\mathrm{Hdim}(X_{k+1}) \geq \mathrm{Hdim}(Y)$ (X_{k+1} could possibly be $X_n \equiv Y$). Then X_{k+1} is part of a mapping cone sequence $A_k \to X_k \to X_{k+1}$ and $\mathrm{Hdim}(X_{k+1}) \geq \mathrm{Hdim}(X_k)$ since X_{k+1} is the first space with $\mathrm{Hdim}(X_{k+1}) \geq \mathrm{Hdim}(Y)$ and $\mathrm{Hdim}(X) < \mathrm{Hdim}(Y)$. By Lemma 26, $\mathrm{Hdim}(A_k) = \mathrm{Hdim}(X_{k+1}) - 1$. Then $\mathrm{Hdim}(A_k) + 1 \geq \mathrm{Hdim}(Y)$ so that $1 - \frac{1}{\mathrm{Hdim}(Y)} \leq 1 - \frac{1}{\mathrm{Hdim}(A_k)+1} \leq \ell^\omega(X,Y)$. □

Lemma 28 *Let $\omega = \omega_T$. If* $\mathrm{Hdim}(Y) < \mathrm{Hdim}(X)$, *then* $\omega(X) \leq l^\omega(X,Y)$.

Proof Let

$$\begin{tikzcd} A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\ X \equiv X_0 \arrow[r] & X_1 \arrow[r] & \ldots \arrow[r] & X_{n-1} \arrow[r] & X_n \equiv Y \end{tikzcd}$$

be any ω-decomposition of X into Y. Let X_{k+1} be the first space with $\mathrm{Hdim}(X_{k+1}) < \mathrm{Hdim}(X)$. Then X_{k+1} is part of a mapping cone sequence $A_k \to X_k \to X_{k+1}$ and $\mathrm{Hdim}(X_{k+1}) < \mathrm{Hdim}(X) \leq \mathrm{Hdim}(X_k)$. By Lemma 26, $\mathrm{Hdim}(A_k) = \mathrm{Hdim}(X_k)$ so that $\omega(X) \leq \omega(X_k) = \omega(A_k) \leq l^\omega(X,Y)$. □

Corollary 29 *Let $\omega = \omega_T$. If $X = \Sigma A$ then* $\mathrm{cl}^\omega(X) = l^\omega(*,X) = \omega(A)$.

Proof The diagram

$$\begin{tikzcd} A \arrow[d] & \\ * \arrow[r] & \Sigma A = X \end{tikzcd}$$

shows that $\mathrm{cl}^\omega(X) \leq \omega(A)$. Now $\mathrm{Hdim}(X) > \mathrm{Hdim}(*) = 0$ so by Lemma 27, $\omega(A) = 1 - \frac{1}{\mathrm{Hdim}(X)} \leq \mathrm{cl}^\omega(X)$ which completes the proof. □

Corollary 30 *Let $\omega = \omega_T$. Then* $\mathrm{kl}^\omega(X) = \omega(X)$.

Proof Clearly $l^\omega(X,*) \leq \omega(X)$. Let $Y = *$ and apply Lemma 28 for the reverse direction. □

Remark 31 Let A be any space and $\mathrm{Hdim}(A) = n$. Then the diagram

$$\begin{array}{ccc} A & & \\ \downarrow & & \\ * & \longrightarrow & \Sigma A = X \end{array}$$

along with Corollary 29 shows that $\mathrm{cl}^\omega(X) = \omega(A) = 1 - \frac{1}{n+1}$. Since $\mathrm{kl}^\omega(X) = \omega(X) = 1 - \frac{1}{n+2}$, we have that $\mathrm{cl}^\omega(X) = 1 - \frac{1}{n+1} < 1 - \frac{1}{n+2} = \mathrm{kl}^\omega(X)$. However, if X is not a suspension, then $\mathrm{kl}^\omega(X) \leq \mathrm{cl}^\omega(X)$ as we now show.

Proposition 32 *Let $\omega = \omega_T$ and $X \not\equiv \Sigma A$ for any A with* $\mathrm{Hdim}(X) = n \geq 1$. *Then* $\mathrm{kl}^\omega(X) \leq \mathrm{cl}^\omega(X)$ *with strict inequality for* $n > 1$.

Proof Let

$$\begin{array}{ccccccccc} & & A_0 & & A_1 & & & & A_{m-1} & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ * \equiv & & X_0 & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X_{m-1} & \longrightarrow & X_m \equiv X \end{array}$$

be any ω-cone length decomposition of $*$ into X. By Lemma 27, $1 - \frac{1}{n} \leq \mathrm{cl}^\omega(X)$. Since $X \not\equiv \Sigma A$ for any A, $\mathrm{cl}(X) \geq 2$ where $\mathrm{cl}(X)$ is the classical notion of cone length. Hence $m \geq 2$ so there is a second attachment, say A_j, in addition to the attachment found in the proof of Lemma 27. The smallest value that A_j can contribute is $\omega(A_j) = 1 - \frac{1}{1+1} = \frac{1}{2}$. Then $1 - \frac{1}{n} + \frac{1}{2} \leq \mathrm{cl}^\omega(X)$. Now $\mathrm{kl}^\omega(X) = \omega(X) = 1 - \frac{1}{n+1}$ by Corollary 30. A simple algebra calculation shows that $1 - \frac{1}{n+1} < 1 - \frac{1}{n} + \frac{1}{2}$ for $n > 1$ and equality for $n = 1$. Thus we have

$$\mathrm{kl}^\omega(X) = 1 - \frac{1}{n+1} < 1 - \frac{1}{n} + \frac{1}{2} \leq \mathrm{cl}^\omega(X)$$

and $\mathrm{kl}^\omega(X) \leq \mathrm{cl}^\omega(X)$ for $n = 1$. $\square$

5.3 $\mathrm{kl}^\omega(X) < \omega(X)$

Since the cone length itself provides an invariant of the complexity of a space X, we define $\omega(X) = \mathrm{cl}(X)$, the classical cone length of X. Consider $\mathrm{kl}^\omega(\mathbb{C}\mathrm{P}^n)$. We clearly have the standard upper bound $\mathrm{kl}^\omega(\mathbb{C}\mathrm{P}^n) \leq n$ from the mapping cone sequence $\mathbb{C}\mathrm{P}^n \xrightarrow{\mathrm{id}} \mathbb{C}\mathrm{P}^n \longrightarrow *$. So far, all the ω-functions we have encountered have satisfied $\mathrm{kl}^\omega(X) = \omega(X)$. For $\omega(X) = \mathrm{cl}(X)$, however, this is not the case as the following Theorem implies:

Theorem 33 *(Cuvilliez, Félix [6]) For X path-connected, we have* $\mathrm{kl}(X) \leq \lceil \log_2(\mathrm{cl}(X) + 1) \rceil$.

It is easy to see that $\lceil \log_2(n+1) \rceil < n$ for $n \geq 3$, so the Theorem implies that $\mathrm{kl}(\mathbb{C}\mathrm{P}^n) \leq \lceil \log_2(n+1) \rceil < n = \omega(\mathbb{C}\mathrm{P}^n)$. Since each attachment is a suspension, each attachment only yields an ω-value of 1, so that $\mathrm{kl}^\omega(\mathbb{C}\mathrm{P}^n) \leq \lceil \log_2(n+1) \rceil$ and in particular $\mathrm{kl}^\omega(\mathbb{C}\mathrm{P}^n) < n$. In fact, since the logarithm grows so slowly, $\mathrm{kl}^\omega(\mathbb{C}\mathrm{P}^n)$ becomes significantly smaller than $\omega(\mathbb{C}\mathrm{P}^n)$ for larger and larger values of n. For example, if $n = 10$, then $\mathrm{kl}^\omega(\mathbb{C}\mathrm{P}^{10}) \leq \lceil \log_2(10+1) \rceil = 4 < 10$. If $n = 100$, then $\mathrm{kl}^\omega(\mathbb{C}\mathrm{P}^{100}) \leq \lceil \log_2(100+1) \rceil = 7 < 100$.

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