Homologically Equivalent Discrete Morse Functions

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Abstract

A theory of homological equivalence of discrete Morse functions is developed in this paper, extending the work of Ayala et al. [1, 3]. We define the homological sequence associated with a discrete Morse function on any finite simplicial complex. This sequence is shown to satisfy specified desirable properties. These properties allow us to show that homological sequences may be viewed as lattice walks satisfying certain parameters. We count the number of discrete Morse functions up to homological equivalence on any collapsible 2-dimensional simplex by constructing discrete Morse functions satisfying certain properties. The paper concludes with an example to illustrate our construction.

Keywords: Discrete Morse theory, Homology, Betti number, Level subcomplex

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1 Introduction

Discrete Morse theory was invented by Robin Forman [6] as an analogue of "smooth" Morse theory popularized by Milnor [9]. Many classical results in Morse theory, such as the Morse inequalities, carry over into the discrete setting

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[8]. Applications of discrete Morse theory are vast, ranging from applications in configuration spaces [10] to computer science search problems [7].

Let f, g be two discrete Morse functions defined on a 1-dimensional simplicial complex, i.e., a graph. Inspired by Nicolaescu [11], R. Ayala et al. [1] studied the homological sequence of a discrete Morse function by introducing the notion of f and g being homologically equivalent, and they counted the number of excellent discrete Morse functions on all graphs [5]. The authors continued their study of homological sequences [4, 3], where in the latter paper they defined the homological sequence on 2-dimensional simplicial complexes. We continue their work in this paper by defining the homological sequence for all finite simplicial complexes. With only a slight loss of generality, we work with essential discrete Morse functions as opposed to excellent discrete Morse functions. We are then able to prove in Theorem 3.4 that the homological sequence of any essential discrete Morse function exhibits the same kind of behaviour as Ayala et al. proved in the 1 and 2 dimensional case. From the properties we prove in Theorem 3.4, it is then immediate that an upper bound for the number of essential discrete Morse functions, with $m = 2k+1$ critical values, on a given collapsible complex of dimension n is the number of lattice walks on $\mathbb{Z}^n$ of length $2k$ that start and end at $(1, 0, 0, \ldots, 0)$ with each value $(a_1, a_2, \ldots, a_n)$ in the walk satisfying $a_i \geq 0$ for all $2 \leq i \leq n$ and $a_1 \geq 1$. It was proved by Nicolaescu [11] that for $n = 2$, the number of such walks is given by $C_k C_{k+1}$ where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number. In fact, Nicolaescu derived this computation while counting the number of *smooth* Morse functions up to homological equivalence on S^2. We develop an alternative formula for this value in Proposition 4.1. We give a construction in Theorem 4.3 to show that when Δ is a collapsible 2-dimensional simplicial complex, we may construct $C_k C_{k+1}$ such discrete Morse functions. Our paper concludes with an example of the construction in Example 4.4.

2 Preliminaries

Let $[n] = \{1, 2, 3, \ldots, n\}$. An *abstract (finite type) simplicial complex* Δ on $[n]$ is a collection of subsets of $[n]$ such that

1. If $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.
2. $\{i\} \in \Delta$ for every $i \in [n]$.

An element $\sigma \in \Delta$ of cardinality $i+1$ is called an *i-dimensional face* or an *i-face* of Δ. A 0-face is sometimes called a *vertex*. If $\sigma, \tau \in \Delta$ with $\tau \subseteq \sigma$, then σ is a *face* of τ and τ is a *coface* of σ. The *dimension* of Δ, denoted $\dim(\Delta)$, is the maximum of the dimensions of all its faces. We use $\sigma^{(i)}$ to denote a simplicial complex of dimension i, and we write $\tau < \sigma^{(i)}$ to denote any subcomplex of σ of dimension strictly less than i. By convention, the empty set $\emptyset$ is the unique simplex of dimension -1 in every simplicial complex.

Definition 2.1 *A discrete Morse function f on Δ is a function $f\colon \Delta \to \mathbb{R}$ such* *that for every p-simplex* $\sigma \in \Delta$*, we have*

$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| \leq 1$$

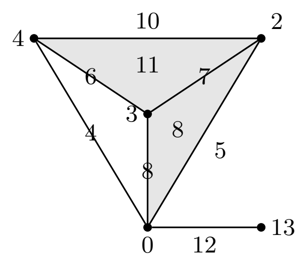
A p-simplex $\sigma \in \Delta$ *is said to be critical with respect to a discrete Morse function* f *if*

$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| = 0$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| = 0.$$

Example 2.2



The above 2*-dimensional simplicial complex is labeled with discrete Morse function* f*. The critical vertices are* $f^{-1}(0), f^{-1}(2)$ *and* $f^{-1}(3)$ *while the critical edges are* $f^{-1}(5), f^{-1}(6), f^{-1}(7)$*, and* $f^{-1}(10)$*. The* 2*-simplex with value* 8 *is not critical while the* 2*-simplex with value* 11 *is critical.*

We say that a discrete Morse function f is *essential* if, given f with m critical values $c_0 < c_1 < \ldots < c_{m-1}$, $f^{-1}(c_i) = \sigma_i$ for some unique critical simplex $\sigma_i \in \Delta$.

Let $c \in \mathbb{R}$. The *level subcomplex* $\Delta(c)$ is the subcomplex of Δ consisting of all simplicies τ with $f(\tau) \leq c$ as well as their faces i.e.,

$$\Delta(c) = \bigcup_{f(\tau) \leq c} \bigcup_{\sigma \leq \tau} \sigma.$$

For each critical value $c_0, c_1, \ldots, c_{m-1}$ of f, we are interested in studying the behaviour of the Betti numbers of the level subcomplexes $\Delta(c_0) \subset \Delta(c_1) \subset \ldots \subset \Delta(c_{m-1})$. We review simplicial homology and Betti numbers below.

2.1 Homological Sequences

We briefly recall the theory of simplicial homology. Since we are only interested in the Betti numbers, we use coefficients in $\mathbb{R}$. Let Δ be a finite-type simplicial complex on $[n]$. Denote by $F_i(\Delta)$ the set of i-dimensional faces of Δ. Let

$\sigma \in F_i(\Delta)$. Then to each σ, we associate the symbol e_σ to represent a basis element in the vector space $k^{|F_i(\Delta)|}$ generated by all the elements of $F_i(\Delta)$. The boundary operators $\partial_i\colon k^{|F_i(\Delta)|} \to k^{|F_{i-1}(\Delta)|}$ are defined as follows: let $\sigma \in F_i(\Delta)$ and define $\partial_i(e_\sigma) = \sum_{j\in\sigma} \operatorname{sgn}(j,\sigma)e_{\sigma-j}$ where $\operatorname{sgn}(j,\sigma) = (-1)^{i-1}$ if j is the i^{th} element of σ when the elements of σ are listed in increasing order. Then $\operatorname{im}(\partial_{i+1}) \subseteq \ker(\partial_{i+1})$, and we define the i^{th} *(unreduced) homology* of Δ to be the vector space $H_i(\Delta) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1}) = k^{\operatorname{nul}(\partial_i)-\operatorname{rank}(\partial_{i+1})}$. The i^{th} *Betti number* of Δ is defined to be $b_i(\Delta) = \operatorname{nul}(\partial_i) - \operatorname{rank}(\partial_{i+1})$. Clearly $b_j(\Delta) = 0$ for $j > n$.

Now let $f\colon \Delta \to \mathbb{R}$ be an essential discrete Morse function on Δ. To each level subcomplex $\Delta(c_i)$, we consider the Betti numbers $b_i(\Delta(c_i))$. The *homological sequence of* f is given by the $n+1$ maps $B_0^f, B_1^f, \ldots, B_n^f\colon \{0, 1, \ldots, m-1\} \to \mathbb{N} \cup \{0\}$ defined by $B_k^f(i) = b_k(\Delta(c_i))$ for all $0 \leq k \leq n$ and $0 \leq i \leq m-1$. We usually write $B_k(i)$ for $B_k^f(i)$ when the discrete Morse function f is clear from the context.

Example 2.3 *Consider the discrete Morse function f in Example 2.2. This is an essential discrete Morse function with critical values* $0, 2, 3, 5, 6, 7, 10,$ *and* 11. *To find the homological sequence of f, we list the Betti numbers of* $\Delta(0), \Delta(2), \Delta(3), \Delta(5), \Delta(6), \Delta(7), \Delta(10),$ *and* $\Delta(11)$. *This is summarized in the following table:*

$B_0:$	1	2	3	2	1	1	1	1
$B_1:$	0	0	0	0	0	1	2	1
$B_2:$	0	0	0	0	0	0	0	0

Notice that only one value changes when moving from column to column and that the last column is the homology of the original simplex Δ even though $\Delta \neq \Delta(11)$. These observations and others are true of the homological sequence of any essential discrete Morse function. We prove this in Theorem 3.4.

Two essential discrete Morse functions $f, g\colon \Delta \to \mathbb{R}$ with m critical values are *homologically equivalent* if $B_k^f(i) = B_k^g(i)$ for all $0 \leq k \leq m-1$ and $0 \leq i$. Homologically equivalent discrete Morse functions were first introduced and studied by Ayala et al. [1]. When Δ is a 1-dimensional simplicial complex, the authors showed the following:

Proposition 2.4 *[1] If f is an essential discrete Morse function on a 1-dimensional simplicial complex, then the homological sequence of f satisfies* $|B_0(i+1) - B_0(i)| = 0, 1$ *and* $B_1(i+1) - B_1(i) = 0, 1$. *In addition, for all* $i = 0, 1, \ldots, m-2$, *exactly one of the following holds:*

1. $B_0(i) = B_0(i+1)$
2. $B_1(i) = B_1(i+1)$.

In Theorem 3.4, we generalize this result to the homological sequence of an essential discrete Morse function on any finite simplicial complex.

3 Homological Sequences

In order to generalize Proposition 2.4 the following lemmas are required, the first of which is a classical result in discrete Morse theory due to Forman.

Lemma 3.1 *[6, Theorem 3.3] If $a < b$ are real numbers such that $[a, b]$ contains no critical values of f, then $b_i(\Delta(a)) = b_i(\Delta(b))$ for all integers $i \geq 0$.*

Lemma 3.2 *Let σ^p be a p-dimensional simplex such that $\sigma^p \notin \Delta$ and $\Delta \cup \sigma^p$ is a simplicial complex. Write $\overline{\Delta} = \Delta \cup \sigma^p$. For every integer $i \geq 0$, exactly one of the following holds:*

1. $b_p(\overline{\Delta}) - b_p(\Delta) = 1$ *and* $b_{p-1}(\overline{\Delta}) - b_{p-1}(\Delta) = 0$
2. $b_{p-1}(\overline{\Delta}) - b_{p-1}(\Delta) = -1$ *and* $b_p(\overline{\Delta}) - b_p(\Delta) = 0$

Furthermore, $b_d(\overline{\Delta}) = b_d(\Delta)$ for all $d \neq p, p-1$.

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of Δ and $\sigma^p \subseteq \{v_1, v_2, \ldots, v_n\}$ be a p-dimensional simplex such that $\sigma^p \notin \Delta$. Let N_p denote the set of p-dimensional faces of Δ. Consider the chain complex

$$\cdots \to k^{|N_{p+1}|} \xrightarrow{\partial_{p+1}} k^{|N_p|} \xrightarrow{\partial_p} k^{|N_{p-1}|} \xrightarrow{\partial_{p-1}} k^{|N_{p-2}|} \to \ldots.$$

The p^{th} and $(p-1)^{st}$ Betti numbers of Δ are defined by $b_p = \text{nul}(\partial_p) - \text{rank}(\partial_{p+1})$ and $b_{p-1} = \text{nul}(\partial_{p-1}) - \text{rank}(\partial_p)$. The corresponding chain complex for $\overline{\Delta}$ is given by

$$\cdots \to k^{|N_{p+1}|} \xrightarrow{\overline{\partial_{p+1}}} k^{(|(N_p)|+1)} \xrightarrow{\overline{\partial_p}} k^{|N_{p-1}|} \xrightarrow{\partial_{p-1}} k^{|N_{p-2}|} \to \ldots.$$

Since members of Δ are closed under taking subsets and $\sigma^p \notin \Delta$, it follows that there does not exist $\tau \in \overline{\Delta}$ with $|\tau| = p+1$ such that $\sigma_p \subseteq \tau$. Thus the additional row of $\overline{\partial_{p+1}}$ corresponding to σ_p is the zero row, and $\text{rank}(\partial_{p+1}) = \text{rank}(\overline{\partial_{p+1}})$.

In addition, $b_p(\overline{\Delta}) = \text{nul}(\overline{\partial_p}) - \text{rank}(\overline{\partial_{p+1}})$ and $b_{p-1}(\overline{\Delta}) = \text{nul}(\partial_{p-1}) - \text{rank}(\overline{\partial_p})$. Now ∂_p is an $|N_{p-1}| \times |N_p|$ matrix and $\overline{\partial_p}$ is an $|N_{p-1}| \times (|N_p|+1)$ matrix. Since $\overline{\partial_p}$ has one more column than ∂_p, we know that either $\text{rank}(\partial_p) + 1 = \text{rank}(\overline{\partial_p})$ or $\text{rank}(\partial_p) = \text{rank}(\overline{\partial_p})$.

Case 1: $\text{rank}(\partial_p) + 1 = \text{rank}(\overline{\partial_p})$.

We have $b_{p-1}(\overline{\Delta}) - b_{p-1}(\Delta) = \text{nul}(\partial_{p-1}) - \text{rank}(\overline{\partial_p}) - \text{nul}(\partial_{p-1}) + \text{rank}(\partial_p) = \text{rank}(\overline{\partial_p}) - \text{rank}(\partial_p) = -1$. Now we show that $b_p = \overline{b_p}$. By the Rank-Nullity theorem, $\text{rank}(\partial_p) + \text{nul}(\partial_p) = N_p$ and $\text{rank}(\overline{\partial_p}) + \text{nul}(\overline{\partial_p}) = N_p + 1$ so $\text{nul}(\partial_p) = \text{nul}(\overline{\partial_p})$. Thus $b_p(\overline{\Delta}) - b_p(\Delta) = \text{nul}(\overline{\partial_p}) - \text{rank}(\overline{\partial_{p+1}}) - \text{nul}(\partial_p) + \text{rank}(\partial_{p+1}) = 0$.

Case 2: $\text{rank}(\partial_p) = \text{rank}(\overline{\partial_p})$.

We have $b_{p-1}(\overline{\Delta}) - b_{p-1}(\Delta) = \text{nul}(\partial_{p-1}) - \text{rank}(\overline{\partial_p}) - \text{nul}(\partial_{p-1}) + \text{rank}(\partial_p) = \text{rank}(\overline{\partial_p}) - \text{rank}(\partial_p) = 0$. Again, by the Rank-Nullity theorem, $\text{rank}(\partial_p) + \text{nul}(\partial_p) = N_p$ and $\text{rank}(\overline{\partial_p}) + \text{nul}(\overline{\partial_p}) = N_p + 1$ so, $\text{nul}(\partial_p) + 1 = \text{nul}(\overline{\partial_p})$. Thus we have $b_p(\overline{\Delta}) - b_p(\Delta) = \text{nul}(\overline{\partial_p}) - \text{rank}(\overline{\partial_{p+1}}) - \text{nul}(\partial_p) + \text{rank}(\partial_{p+1}) = \text{nul}(\overline{\partial_p}) - \text{nul}(\partial_p) = 1$.

Hence, adding a p-simplex will either decrease b_{p-1} or increase b_p while leaving the other constant. It is clear that $b_d(\overline{\Delta}) = b_d(\Delta)$ for all $d \neq p, p-1$. □

Lemma 3.3 *Let Δ be a simplicial complex with essential discrete Morse function f and suppose that f has global minimum a. Then there is a unique 0-critical simplex σ such that $f(\sigma) = a$.*

Proof By Lemma 3.1, $\Delta(x) = \emptyset$ for all $x < a$. Since f is essential, there exists a unique simplex σ such that $f(\sigma) = a$. Thus $\Delta(a) = \{\sigma\}$ and $|\Delta(a)| - |\Delta(x)| = 1$ so that σ must be a 0-dimensional critical simplex. □

We are now ready to prove our main result. This result can be interpreted as saying that the homological sequence of any essential discrete Morse function is "well-behaved" in the sense that only one Betti number can change for each subsequent level subcomplex by a value of ± 1.

Theorem 3.4 *Let f be an essential discrete Morse function on a connected n-dimensional simplicial complex Δ with m critical values $c_0, c_1, \ldots, c_m$. Then, each of the following holds:*

1. *$B_0(0) = B_0(m-1) = 1$ and $B_d(0) = 0$ for all $d \in \mathbb{Z}^{\geq 1}$*
2. *For all $0 \leq i < m-1$, $|B_d(i+1) - B_d(i)| = 0$ or 1 whenever $0 \leq d \leq n$ and $B_d(i) = 0$ whenever $d \geq n+1$*
3. *$B_d(m-1) = b_d(\Delta)$*
4. *For each $i = 0, 1, \ldots, m-2$ either:*
   - *(a) $B_{p-1}(i) = B_{p-1}(i+1)$*
   - *(b) $B_p(i) = B_p(i+1)$*

   where $p = \dim(f^{-1}(c_i))$. Furthermore $B_d(i) = B_d(i+1)$ for any $d \neq p, p-1$ and $1 \leq d \leq n$

Proof We proceed in order. For 1, choose $y \in \mathbb{N}$ such that $\Delta(c_{m-1} + y) = \Delta$. By Lemma 3.1, $b_0(\Delta_{c_{m-1}}) = b_0(\Delta(c_{m-1} + y)) = b_0(\Delta)$. Since Δ is connected $b_0(\Delta(c_{m-1})) = B_0(m-1) = 1$. By the Lemma 3.3, $\Delta(0) = \sigma_0$. Thus $B_d(0) = 0$ for all $d \in \mathbb{Z}^{\geq 1}$. This proves the first assertion.

For 2, we note that by Lemma 3.1, $b_d(\Delta(c_i)) = b_d(\Delta(x))$ for any $x \in [c_i, c_{i+1})$. Since f is essential, there exists $\epsilon > 0$ such that $\Delta(c_{i+1}) = \Delta(c_{i+1} -$

$\epsilon) \cup \sigma^p$ where σ^p is a critical p-simplex such that $f(\sigma^p) = c_{i+1}$. We now apply Lemma 3.2 for each of the following cases: if $p = d$ then $B_d(i+1) - B_d(i) = 0$ or 1. If $p = d+1$ then $B_d(i+1) - B(i) = -1$ or 0. Otherwise, $B_d(i+1) - B_d(i) = 0$. This proves 2.

For 3, observe that $m - 1$ is the maximum critical value. By Lemma 3.1, B_d is constant for all values $x > c_{m-1}$. Since there is a $y \in \mathbb{N}$ such that $\Delta(c_{m-1} + y) = \Delta$, we see that $B_d(m-1) = b_d(\Delta)$.

Finally, we apply Lemma 3.1 to see that $b_d(\Delta(c_i)) = b_d(\Delta(x))$ for all $x \in [c_i, c_{i+1})$. Since f is essential, there exists $\epsilon > 0$ such that $\Delta(c_{i+1}) = \Delta(c_{i+1} - \epsilon) \cup \sigma_p$ as in the proof of 2. Observe that, by Lemma 3.2, the addition of a p-dimensional simplex will change either B_p or B_{p-1}, leaving all others values fixed. $\square$

4 Counting Discrete Morse Functions

It is easily seen that the result of Theorem 3.4 can be viewed as walks of length $2k$ in the lattice $\mathbb{Z}^n$ that start and end at $(1, 0, 0, \ldots, 0)$ with each value $(a_1, a_2, \ldots, a_n)$ in the walk satisfying $a_i \geq 0$ for all $2 \leq i \leq n$ and $a_1 \geq 1$. If Δ is a 1-dimensional and collapsible (i.e., a tree), then Ayala et al.'s 2009 result [1, Theorem 6.1] shows that the number of homological sequences with $m = 2k+1$ critical values is given by the k^{th} Catalan number.

We investigate the case where Δ is a 2-dimensional collapsible simplicial complex. In this case, we are interested in counting the number of lattice walks of length $2k$ in $\mathbb{Z}^2$ starting and ending at $(1, 0)$ with first coordinate positive and second coordinate nonnegative. This value has been computed explicitly by Nicolaescu [11] to be $C_k C_{k+1}$, the product of consecutive Catalan numbers. We obtain an alternative formula for this value.

Proposition 4.1 *Let f be an essential discrete Morse function on a 2-dimensional collapsible complex with $m = 2k+1$ critical values. Let $j = 2\ell$ for $\ell \in \mathbb{N} \cup \{0\}$ with $j \leq 2k$. The number of homology equivalence classes of essential discrete morse functions is*

$$\sum_{\ell=0}^{k} \binom{m-1}{j} C_{k-\ell} C_{\ell}.$$

Proof It is known for a tree that the number of homological equivalence classes is the k^{th} Catalan number, $C_k = \frac{1}{k+1}\binom{2k}{k}$. By the Theorem 3.4, $B_0(0) = B_0(m-1) = 1$ and $B_1(0) = B_1(m-1) = 0$. Thus every created cycle must eventually be covered, causing the number of times B_1 changes (increases or decreases) to be even. We denote this number by $j = 2\ell$. Next observe that if B_1 changes, B_0 remains constant by Theorem 3.4.

Thus we are left with $m - 1$ positions to place j changes in B_1 giving us $\binom{m-1}{j}$. Observe that the B_1 sequence exhibits a walk of length $\mathbb{Z}^{\geq 0}$ starting and ending at 0 with ℓ steps of size ± 1.

Recall that when B_1 changes B_0 remains constant. Thus there are j fewer critical values left to B_0. We have $m - j = 2(k - l) + 1$ so that there are C_{k-l} arrangements for B_0.

Thus the total number of possible sequences for m critical values is

$$\sum_{\ell=0}^{k} \binom{m-1}{j} C_{k-\ell} C_{\ell}$$

which is what we desired to show. □

Corollary 4.2 $C_k C_{k+1} = \sum_{\ell=0}^{k} \binom{m-1}{j} C_{k-\ell} C_{\ell}$

We now show that the above upper bound is the actual number of essential discrete Morse functions on a collapsible 2-dimensional simplicial complex up to homological equivalence. As noted, for $m = 2k + 1$ critical values, this yields $C_k C_{k+1}$. Nicolaescu computed this value to count the number of smooth Morse functions on the 2-sphere S^2 up to homological equivalence. Hence the Theorem below provides another nice symmetry between smooth and discrete Morse theory. Ayala et al. [3, Theorem 5.1] give a sketch of a proof which counts the number of essential discrete Morse functions on all compact orientable surfaces. Below we give the full details of an alternative proof using those same authors' construction of essential discrete Morse functions on any 1-dimensional simplicial complex [5, Theorem 4.3]. The technical hypothesis about Δ^1 is to ensure that we can apply Ayala's construction to obtain any homological sequence (See introductory remarks of the proof of Theorem 4.4.3 [5]).

Theorem 4.3 *Let Δ be a collapsible 2-dimensional simplicial complex such that Δ^1 contains at least one vertex of degree 1 or that Δ^1 is a non-trivial bridgeless graph. Then the number of essential discrete Morse functions with $m = 2k + 1$ critical elements on Δ is $C_k C_{k+1}$.*

Proof By Corollary 4.2, we know that the number of essential discrete Morse functions on Δ with $m = 2k + 1$ critical values is bounded above by $C_k C_{k+1}$. Hence, given a homological sequence, we construct an essential discrete Morse function with $m = 2k + 1$ critical values. Observe that for some $0 \leq \ell \leq k$, there are 2ℓ nonzero entries in the row B_1 since each time B_1 increases it must eventually decrease to end at 0. By Theorem 3.4, any such homological sequence on Δ is of the form

$$\begin{array}{lcccccccccccc} B_0: & n_0 & n_1 & \dots & n_{t_1} & n_{t_1} & n_{t_1+1} & \dots & n_{t_{2\ell}} & n_{t_{2\ell}} & n_{t_{2\ell+1}} & \dots & 1 \\ B_1: & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0. \end{array}$$

We will construct an essential discrete Morse function g on Δ with the above homological sequence. This is accomplished by constructing an essential discrete

Morse function on Δ^1, the 1-skeleton of Δ. The homological sequence we wish to place on Δ^1 is based on the given homological sequence.

We begin by subdividing Δ as necessary to obtain enough simplicies. Remove the 2-simplicies of Δ to obtain Δ^1. The resulting skeleton is a graph with $b := b_1(\Delta^1)$ independent cycles. By Ayala et al. [3, Theorem 5.1] there is an essential discrete Morse function f on Δ^1 with the following homological sequence:

$$\begin{array}{cccccccccccccc} B_0: & n_0 & n_0 & n_0 & n_0 & \dots & n_0 & n_1 & n_2 & \dots & n_{t_1} & n_{t_1} & \dots & 1 \\ B_1: & 0 & 1 & 2 & 3 & \dots & b-\ell & b-\ell & b-\ell & \dots & b-\ell & b-\ell+1 & \dots & b. \end{array}$$

Since Δ is collapsible, each removed 2-simplex τ may be associated with a critical edge bounding τ. The above sequence is described as follows: start with the desired homological sequence and insert $b-\ell$ holes after the initial critical value. Then the B_0 sequence is shifted $b-\ell$ entries to the right, and the corresponding B_1 value is $b-\ell$ as opposed to 0. The first time B_1 increases in the original sequence is at n_{t_1}. Our new sequence also has B_1 increase after n_{t_1} except that it increases from $b-\ell$ to $b-\ell+1$, whereas the original sequence increases from 0 to 1 at this stage. Continue in this manner until the first time B_1 decreases in the original sequence. This step is ignored in the new sequence (we will return to this below when constructing g).

We define $g = f$ on Δ^1. We will next insert the 2-simplicies back into Δ^1 and label them so that the sequence on Δ^1 is transformed into the desired sequence on Δ. This is accomplished as follows: as stated above, there is a one-to-one correspondence between the first $b-\ell$ critical edges and a 2-simplex that the edges bound. Call these critical edges $e_1, e_2, \dots, e_{b-\ell}$ and their corresponding 2-simplicies $d_1, d_2, \dots, d_{b-\ell}$. Define $g(d_i) = f(e_i), 1 \leq i \leq b-\ell$. Since f is essential, the critical edge e_i is the only simplex with value $f(e_i)$. Hence, defining $g(f_i) = f(e_i)$ will still yield a discrete Morse function, but now each of the e_i is not critical under the function g. In addition, each d_i is non-critical.

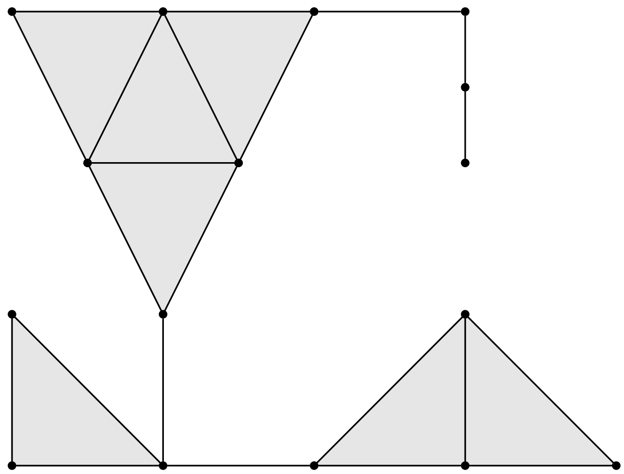
Now let t_j be the first index in the original sequence such that $B_1(t_j) - B_1(t_{j+1}) = 1$. Choose any 2-simplex that has not yet been labeled and whose boundary is in the current level subcomplex. Label this simplex so that it has a value greater than the maximum of all values of the current level subcomplex, but less than the values on $f(\Delta^1)$ that are not yet in the current level subcomplex. This will ensure that the 2-simplex is critical, and thus B_1 will decrease exactly when desired. Repeat this step as necessary.

The resulting discrete Morse function g on Δ will have the given homological sequence. □

4.1 Example

We give an example of the construction in Theorem 4.3.

Example 4.4 *Let Δ be the collapsible 2-dimensional complex given by the following figure:*



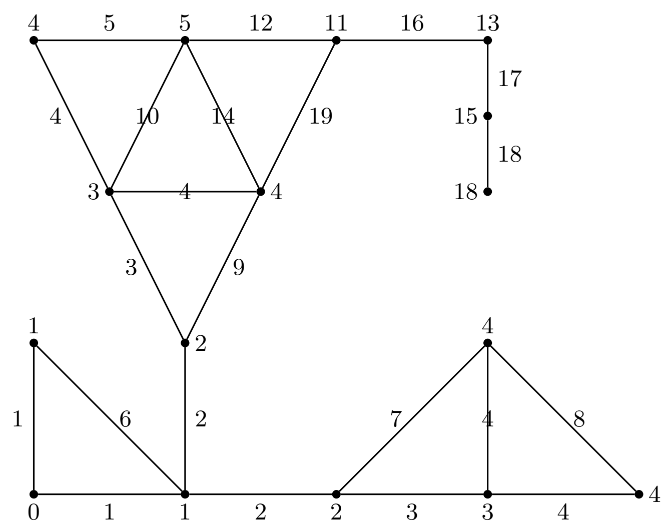
We will construct an essential discrete Morse function on Δ with the following homological sequence:

B_0	1	2	1	2	2	3	3	2	1	1	1
B_1	0	0	0	0	1	1	0	0	0	1	0.

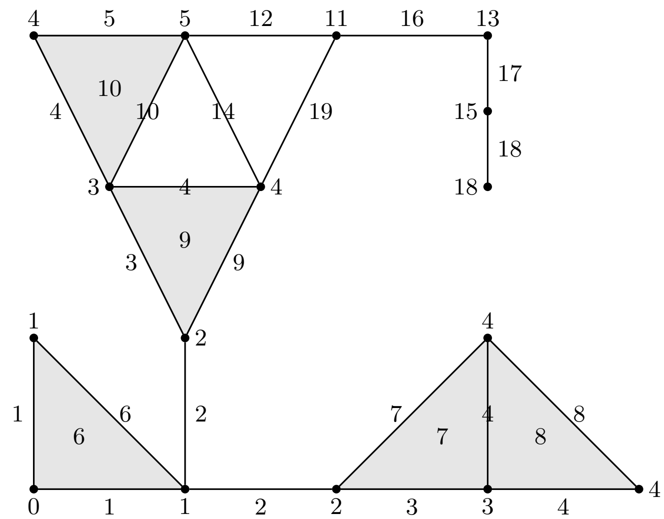
Use the result of Ayala et al. [5, Theorem 4.3] to construct an essential discrete Morse function f on Δ^1 with the following homological sequence:

B_0	1	1	1	1	1	1	2	1	2	2	3	2	1	1
B_1	0	1	2	3	4	5	5	5	5	6	6	6	6	7.

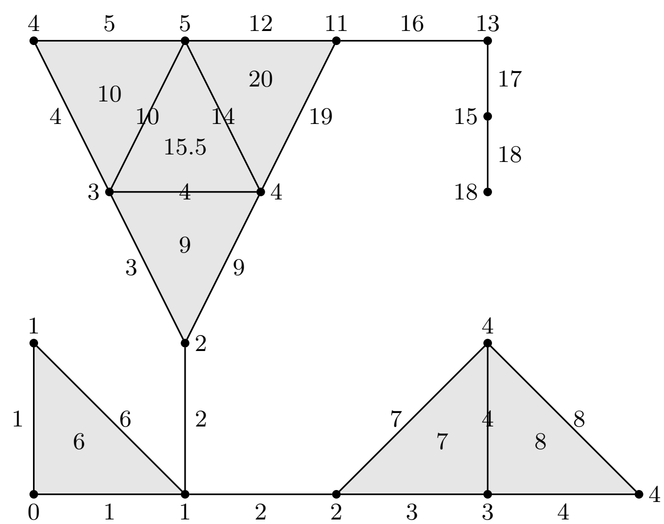
Such a discrete Morse function is given below.



Now pick five 2-simplicies and label them with the maximum value of their boundary edge value.



The remaining two 2-simplicies are labeled slightly greater than the maximum of all the simplicies in the current level subcomplex, where the current level subcomplex is determined by when B_1 decreases in our original homological sequence. Hence the discrete Morse function g is given as follows:



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