# LUSTERNIK–SCHNIRELMANN CATEGORY FOR SIMPLICIAL COMPLEXES

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Abstract. The discrete version of Morse theory due to Robin Forman is a powerful tool utilized in the study of topology, combinatorics, and mathematics involving the overlap of these fields. Inspired by the success of discrete Morse theory, we take the first steps in defining a discrete version of the Lusternik–Schnirelmann category suitable for simplicial complexes. This invariant is based on collapsibility as opposed to contractibility, and is defined in the spirit of the geometric category of a topological space. We prove some basic results of this theory, showing where it agrees and differs from that of the smooth case. Our work culminates in a discrete version of the Lusternik–Schnirelmann theorem relating the number of critical points of a discrete Morse function to its discrete category.

## 1. Introduction

In his landmark paper *Morse theory for cell complexes* [5], Robin Forman presented a new version of Morse theory suitable for cell complexes. This has come to be known as discrete Morse theory. Many of the fundamental theorems in smooth Morse theory have a discrete analogue. For example, the weak and strong Morse inequalities have discrete versions relating the number of critical simplices of a complex $K$ to the Betti numbers of $K$ [7, Theorem 2.11]. It is applied in a variety of settings to study problems concerning collapsibility of certain complexes [13], configuration spaces [16], and computer science search problems [6].

Inspired by the success of a discrete version of Morse theory, we introduce a discrete version of the Lusternik–Schnirelmann category for simplicial complexes. This topological invariant measures the complexity of a space by breaking it up into simpler pieces. Formally, the Lusternik–Schnirelmann category, or L–S category of a topological space $X$, denoted $\mathrm{cat}(X)$, is the minimum number of open sets in a cover of $X$ such that each open set is contractible to a point in $X$. It was originally defined by Lusternik and Schnirelmann in their study of critical points on manifolds [14]. They obtained

1991 *Mathematics Subject Classification.* Primary 55M30, 55U05. Secondary 57M15.

The authors are grateful to Professor Frank Morgan for suggesting this project and to an anonymous referee for helpful comments and suggestions.

what is now known as the Lusternik–Schnirelmann Theorem [3, Theorem 1.15] which states that if $M$ is a smooth manifold satisfying additional properties and $f\colon M \to \mathbb{R}$ is smooth, then $\mathrm{cat}(M) + 1 \leq m$ where $m$ is the number of critical points of $f$. The origins of L–S category in critical point theory combined with our above discussion of the importance of discrete Morse theory sets an immediate precedence for a discrete version of L–S category.

After the original paper by Lusternik and Schnirelmann, many other invariants were defined in order to be able to estimate the L–S category. One such invariant, introduced by Ralph Fox [9], is the geometric category of $X$, denoted $\mathrm{gcat}(X)$. Rather than consider the minimum number of open sets in a cover of $X$ such that each open set is contractible to a point in $X$, Fox removed the condition that each set in the cover had to be contractible in $X$ and just required that they be contractible. It is in the spirit of this geometric category that we make our definition on a simplicial complex. While one could define category of a simplicial complex as the category of its geometric realization, we seek a definition that can be studied in the world of simplicial complexes. Inspired by J. H. C. Whitehead's theory of simple homotopy type [2, 19, 20], we use "collapsible" as our measure of simplicity and study the minimum number of collapsible sets in $K$ that it takes to cover $K$. To avoid confusion, we use the term smooth category or smooth geometric category when discussing the classical L–S and geometric category. It turns out that when $K$ is a 1-dimensional complex (i.e. a graph), our definition coincides with arboricity of a graph, defined by Nash-Williams [17]. Another immediate interesting connection is given in Proposition 11 where the discrete category is related to the Zeeman conjecture and by extension the Poincaré conjecture. We compare the discrete category to the smooth L–S category in Proposition 12, showing that the former always bounds above the geometric realization of the latter. In Example 21, we see that not only can inequality be strict, but the discrete category of a space can be arbitrarily larger than its dimension as a simplicial complex. This is in stark contrast to the smooth geometric category which is always bounded above by its dimension. We deduce some fundamental properties of the discrete category in Section 3.2, largely inspired by results for the smooth category. In Section 3.5, we prove a discrete version of the celebrated Lusternik–Schnirelmann Theorem.

## 2. Prelimnaries

All spaces and simplicial complexes are assumed to be connected.

### 2.1. Simplicial Complexes.

**Definition 1.** Let $[n] = \{1, 2, 3, \ldots, n\}$. An **abstract (finite type) simplicial complex** $K$ on $[n]$ is a collection of subsets of $[n]$ such that

(1) If $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.

(2) $\{i\} \in K$ for every $i \in [n]$.

An element $\sigma \in K$ of cardinality $i+1$ is called an **$i$-dimensional face** or an **$i$-face** of $K$. The **dimension** of $K$, denoted $\dim(K)$, is the maximum of the dimensions of all its faces. If $\sigma, \tau \in K$ with $\sigma \subseteq \tau$, then $\sigma$ is a **face** of $\tau$ and $\tau$ is a **coface** of $\sigma$. A face of $K$ that is not contained in any other face is called a **facet** of $K$. By convention, the empty set $\emptyset$ is the unique simplex of dimension $-1$ in every simplicial complex. A **(closed) subcomplex** $L$ of $K$, denoted $L \subseteq K$, is a subset $L$ of $K$ such that $L$ is also a simplicial complex. If $\sigma \in K$, we write $\overline{\sigma}$ to denote the closed subcomplex generated by $\sigma$. The **$i$-skeleton** of $K$ is given by $K^i = \{\sigma \in K : \dim(\sigma) \leq i\}$.

Two fundamental constructions in topology, the cone and the suspension, are special cases of the **join** of two simplicial complexes.

**Definition 2.** Let $K, K'$ be two simplicial complexes. The **join** of $K$ and $K'$ is defined by $K * K' := \{\{x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m\} : \{x_0, x_1, \ldots, x_n\} \in K, \{y_0, y_1, \ldots, y_m\} \in K'\}$. If $K' = \{v\}$ is given by a single point, we write $CK := K * K'$ for the **cone** on $K$. The vertex $v \in CK$ is called the **apex of the cone**. If $K' = \{v, u\}$, we write $\Sigma K := K * K'$ for the **suspension** of $K$.

2.2. **Geometric Realization.** If $K$ is a simplicial complex, let $|K|$ denote its geometric realization. We note here some nice features of the geometric realization.

**Proposition 3.** *Let $K$ be a simplicial complex. Then the simplicial homology groups $H_*(K)$ of $K$ are isomorphic to $H_*(|K|)$, the simplicial homology of its geometric realization. Furthermore, $|K|$ is a CW complex.*

We now introduce collapses, the building block of the discrete category.

**Definition 4.** Let $K$ be a simplicial complex and suppose that there is a pair of simplices $\sigma, \tau$ such that $\sigma$ is a face of $\tau$ and $\sigma$ has no other cofaces. Then $K - \{\sigma, \tau\}$ is a simplicial complex called an **elementary (simplicial) collapse** of $K$, and $K$ is said to **collapse** onto $L$ if $L$ can be obtained from $K$ through a finite series of elementary collapses, denoted $K \searrow L$. If $K$ collapses to $L$, we also say that $L$ **expands** to $K$, denoted $L \nearrow K$. We say that $K$ and $L$ are of the same **simple homotopy type**, denoted $K \sim L$. In the case where $L = \{v\}$ is a single point, we say that $K$ is **collapsible**. More generally, let $L, L' \subseteq K$ be simplicial complexes. A sequence of elementary collapses and expansions $L = L_0 \to L_1 \to \ldots \to L_n = L'$ such that $L_i \subseteq K$ for all $0 \leq i \leq n$ is a **simple homotopy equivalence between $L$ and $L'$ in $K$**. We write $L \sim_K L'$.

If $K$ and $L$ have the same simple homotopy type, then there is a sequence of elementary collapses and expansions $K = K_0 \to K_1 \to \ldots \to K_m = L$. If $K_i \to K_{i+1}$ is an elementary collapse, then $|K_{i+1}|$ is a deformation retraction of $|K_i|$. Similarity, if $K_j \to K_{j+1}$ is an elementary expansion, then $|K_j|$ is a deformation retraction of $|K_{j+1}|$. This yields the following:

**Proposition 5.** [12, Proposition 6.14] *If $K$ and $L$ have the same simple homotopy type, then $|K|$ and $|L|$ have the same homotopy type. In particular, if $K$ is collapsible, then $|K|$ is contractible.*

2.3. **Geometric Category.** The discrete category is more analogous to that of the geometric category than the classical Lusternik–Schnirelmann category. For example, Fox showed that the geometric category is not a homotopy invariant, while Example 14 shows that the discrete category is not a simple homotopy invariant. See Chapter 3 of the book [3] for more information on the geometric category. Here we introduce the geometric category and prove a proposition which will allow us to compare the discrete and smooth versions of geometric category.

**Definition 6.** Let $X$ be a CW complex. The **(open) geometric category** of $X$, denoted $\mathrm{gcat}^{\mathrm{op}}(X)$ or just $\mathrm{gcat}(X)$, is the least integer $n$ such that there exists an open cover $\{U_0, U_1, \ldots, U_n\}$ of $X$ with each $U_i$ contractible. The **(closed) geometric category** of $X$, denoted $\mathrm{gcat}^{\mathrm{cl}}(X)$, is the least integer $n$ such that there exists a closed cover $\{U_0, U_1, \ldots, U_n\}$ of $X$ with each $U_i$ a contractible subcomplex of $X$.

If $\overline{\sigma} \subseteq K$ is a closed subcomplex of the simplicial complex $K$, then $|\overline{\sigma}|$ is a compact subspace [4, Theorem II.2.9)] of a metric space and hence closed. Thus any closed cover of $K$ corresponds to a closed cover of $|K|$. The following proposition will relate the discrete geometric category of $K$ to the geometric category of its geometric realization.

**Proposition 7.** *Let $X$ be a CW complex. Then* $\mathrm{gcat}^{\mathrm{op}}(X) \leq \mathrm{gcat}^{\mathrm{cl}}(X)$.

*Proof.* Let $\{U_0, U_1, \ldots, U_n\}$ be a contractible cover of $X$ with each $U_i$ a closed subcomplex of $X$. Then for each $U_i$, there is an open set $V_i$ containing $U_i$ with $U_i$ a strong deformation retraction of $V_i$ [18, Theorem 8.29]. The contraction of $U_i$ composed with the strong deformation provides a contraction of each $V_i$. Thus $V_i$ is contractible and $\{V_0, V_1, \ldots, V_n\}$ covers $X$ so that $\mathrm{gcat}^{\mathrm{op}}(X) \leq \mathrm{gcat}^{\mathrm{cl}}(X)$. □

## 3. Discrete Geometric category

We introduce our main focus of study, the discrete geometric category of a simplicial complex.

3.1. **Definition.**

**Definition 8.** Let $L \subseteq K$ be a subcomplex. We say that $L$ **has discrete geometric pre-category less than or equal to $m$ in $K$**, denoted $\widetilde{\mathrm{dgcat}}_K(L) \leq m$, if there exists $m+1$ closed subcomplexes $\{U_0, U_1, \ldots, U_m\}$, $U_i \subseteq K$ for

$0 \leq i \leq m$, each of which is collapsible such that $L \subseteq \bigcup_{i=0}^{m} U_i$. If $\widetilde{\mathrm{dgcat}}_K(L) \not< m$, then $\widetilde{\mathrm{dgcat}}_K(L) := m$. The **discrete (geometric) category** of $L$ in $K$ is defined by $\mathrm{dgcat}_K(L) := \min\{\widetilde{\mathrm{dgcat}}_K(L') : L \text{ collapses to } L'\}$. We write $\mathrm{dgcat}(K) := \mathrm{dgcat}_K(K)$. It follows immediately from the definition that if $L \searrow L'$ then $\widetilde{\mathrm{dgcat}}_K(L) \geq \widetilde{\mathrm{dgcat}}_K(L')$ and $\mathrm{dgcat}_K(L) \leq \widetilde{\mathrm{dgcat}}_K(L)$.

*Remark* 9. A word is in order concerning our definition. The need to define the pre-category of a complex is in order to guarantee that an elementary collapse does not decrease the discrete category. Our definition of pre-category avoids the necessity to subdivide, which is another possible approach to avoiding this problem.

The discrete geometric category generalizes the notion of a collapsible simplicial complex, as the following proposition demonstrates.

**Proposition 10.** *Let $K$ be a simplicial complex. Then $\mathrm{dgcat}(K) = 0$ if and only if $K$ is collapsible.*

*Proof.* Suppose $\mathrm{dgcat}(K) = 0$. Then there exists a subcomplex $L \subseteq K$ such that $K \searrow L$ and $\widetilde{\mathrm{dgcat}}(L) = 0$. Since $\widetilde{\mathrm{dgcat}}(L) = 0$, there is a sequence of elementary collapses such that $L \searrow \{v\}$ for some vertex $v \in L$. Combining the collapses $K \searrow L$ and $L \searrow \{v\}$, we obtain a collapse of $K \searrow \{v\}$, and hence $K$ is collapsible. The other direction is immediate. □

In 1963, Zeeman conjectured that if $K$ is a contractible 2-dimensional complex then $K \times I$ is collapsible. In the same paper, he also proved that the the Zeeman conjecture implies the Poincaré conjecture [21].

Combining this with Proposition 10, we obtain a rephrasing of the Zeeman conjecture in terms of discrete geometric category.

**Proposition 11.** *Let $K$ be a 2-dimensional contractible complex. Then $K \times I$ is collapsible if and only if $\mathrm{dgcat}(K \times I) = 0$. In particular, if $\mathrm{dgcat}(K \times I) = 0$ then the Poincaré conjecture holds.*

3.2. **Basic Properties.** It is now easy to show the relationship between discrete geometric category and geometric category.

**Corollary 12.** $\mathrm{gcat}(|K|) \leq \mathrm{dgcat}(K)$.

*Proof.* The geometric realization of a (closed) subsimplex of $K$ is a union of closed subcomplexes of $|K|$ [10, Theorem 3.3.2 and following discussion]. This fact combined with Proposition 5 shows that a collapsible cover of $K$ is sent to a contractible closed cover of $|K|$. Hence $\mathrm{gcat}^{\mathrm{op}}(|K|) \leq \mathrm{gcat}^{\mathrm{cl}}(|K|) \leq \mathrm{dgcat}(K)$ by Proposition 7. □

We show below that this inequality can be strict. Since cohomology of simplicial complexes is well studied, the following corollary yields a rudimentary lower bound for the discrete category based on its relationship to the smooth category. We also prove a simple upper bound.

**Proposition 13.** *If $K$ is a simplicial complex on $n$ vertices, then* $\cup(|K|) \leq \mathrm{dgcat}(K) \leq n-1$.

*Proof.* Combining [3, Proposition 1.5], the fact that $\mathrm{cat}(X) \leq \mathrm{gcat}(X)$, and Corollary 12, we see that $\cup(|K|) \leq \mathrm{dgcat}(K)$. To show the upper bound, let $v \in K$ be a 0-simplex,and define $L_v = \overline{\{\sigma \in K : v \in \sigma\}}$. We claim that $\{L_v\}_{v \in K}$ is a collapsible cover of $K$. To see that $L_v$ is collapsible, let $\sigma \in L_v$ be a face of maximum dimension in $L_v$. Then $\sigma - \{v\} \in L_v$ is a free face of $\sigma$ since any other coface of $\sigma - \{v\}$ is of the form $\sigma - \{v\} \cup \{x\} \notin L_v$. Thus $L_v \searrow L_v - \{\sigma, \sigma - \{v\}\}$. Now we take $\tau \in L_v - \{\sigma, \sigma - \{v\}\}$ to be a face of maximum dimension in $L_v - \{\sigma, \sigma - \{v\}\}$ and repeat to collapse $L_v$ to $v$. Since $\{L_v\}_{v \in K}$ clearly covers $K$, $\mathrm{dgcat}(K) \leq n-1$. □

*Example* 14. It is well known that there are simplicial complexes which have contractible geometric realization but which are not collapsible, such as Bing's house with two rooms [1] and the dunce cap $D$. These examples show that the inequality in Corollary 12 can be strict. One might ask how large the difference can be. In Example 21, we will show that the difference $\mathrm{dgcat}(K) - \mathrm{gcat}(|K|)$ can be made arbitrarily large, even for $K$ a 1-dimensional simplicial complex. However, it is also the case that $D \sim \{v\}$ but $\mathrm{dgcat}(D) = 1 > 0 = \mathrm{dgcat}(\{v\})$. Hence dgcat is not a simple homotopy invariant.

**Proposition 15.** *Let $K, L \subseteq K \cup L$. Then* $\mathrm{dgcat}(K \cup L) \leq \mathrm{dgcat}(K) + \mathrm{dgcat}(L) + 1$.

*Proof.* Let $\{U_0, \ldots, U_n\}$ be a collapsible cover of $K$ and $\{V_0, \ldots, V_m\}$ a collapsible cover of $L$. Clearly $\{U_0, \ldots, U_n, V_0, \ldots, V_m\}$ covers $K \cup L$ so that $\mathrm{dgcat}(K \cup L) \leq n + m + 1$. □

3.3. **Examples.** As in the smooth case, the discrete category is 0 on a cone and at most 1 for a suspension.

**Proposition 16.** *Let $K$ be a simplicial complex. Then* $\mathrm{dgcat}(CK) = 0$.

*Proof.* We show that $CK$ collapses to a point. Let $v \in CK$ be the apex and write $\sigma_v = \sigma \cup \{v\}$ for any $\sigma \in K$ and view $K \subseteq CK$ as the base of the cone. Assume $\sigma$ is an $n$-dimensional facet of $K$. Then $\sigma$ is the unique coface of $\sigma_v$ so that $CK \searrow CK - \{\sigma, \sigma_v\}$. We may remove every $n$-dimensional facet of $K$ in this way so that the $(n-1)$-dimensional faces are now the facets of the base. Repeat this process. Since $\sigma$ is in 1-1 correspondence with $\sigma_v$, $CK \searrow v$ through a series of elementary collapses. Thus $CK$ is collapsible and $\mathrm{dgcat}(CK) = 0$. □

**Corollary 17.** *Let $K$ be a simplicial complex. Then* $\mathrm{dgcat}(\Sigma K) \leq 1$.

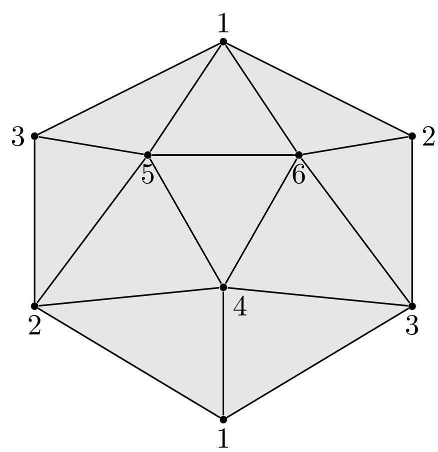
*Proof.* This follows from the fact that $\Sigma K = CK \cup CK$. $\square$

*Example* 18. We show that $\operatorname{dgcat}(S^n) = 1$. Let $\mathcal{P}([n])$ denote the power set of $[n]$. Observe that $S^n = CS^{n-1} \cup \Delta^n$ where $\Delta^n$ is the standard $n$-simplex. This follows from the fact that

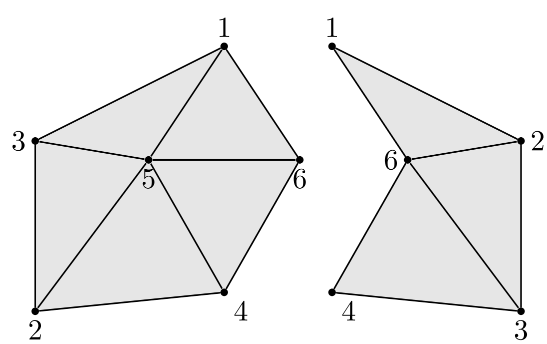
$$\begin{aligned} CS^{n-1} \cup \{1, 2, \ldots, n+1\} &= C(\mathcal{P}([n+1]) - \{1, 2, \ldots, n+1\}) \cup \{1, 2, \ldots, n+1\} \\ &= \mathcal{P}([n+1]) - \{1, 2, \ldots, n+2\} - \{1, 2, \ldots, n+1\} \\ &\quad \cup \{1, 2, \ldots, n+1\} \\ &= S^{n+1}. \end{aligned}$$

Since both $CS^{n-1}$ and $\Delta^n$ are collapsible, $\operatorname{dgcat}(S^n) \leq 1$. Since $S^n$ does not contain any free faces, $S^n$ is not collapsible.

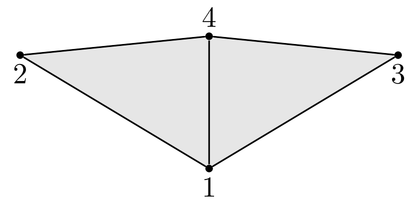
*Example* 19. There is a simplicial structure on $\mathbb{RP}^2$ given by




It is easily seen that the following forms a collapsible cover:



and




It is well known that $\cup(\mathbb{R}\mathrm{P}^2) = 2$ so that by Proposition 13, $\mathrm{dgcat}(\mathbb{R}\mathrm{P}^2) = 2$.

3.4. **1-dimensional complexes.** We focus our attention on the connection between the discrete category of 1-dimensional simplicial complexes (a graph) and a well-known graph theoretic invariant. Nash-Williams [17] defined the arboricity of a graph $G$ to be the minimum number of forests into which the edges of $G$ can be partitioned. This is equivalent to our definition, as the following proposition implies.

**Proposition 20.** *A 1-dimensional simplicial complex is collapsible if and only if it is a tree.*

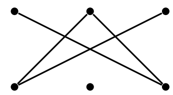
Nash-Williams computed the arboricity for all graphs, and there are many other results known about the arboricity of graphs [11]. For completeness, however, we give our own argument to estimate the arboricity or dgcat of a certain collection of graphs below in order to compare the discrete category with the classical smooth category. In the case of a graph, the geometric category is bounded above by 1 and thus of little interest. However, the discrete category can take on arbitrarily large values on a graph.

*Example* 21. Let $K = K_n$, the complete graph in $n$ vertices. We show that $\mathrm{dgcat}(K) \geq \lceil \frac{n}{2} \rceil - 1$ so that the discrete category of even a 1-dimensional simplicial complex can be made arbitrarily large. If $U_i$ is collapsible, the maximum number of edges that $U_i$ can have is $n-1$. This follows from the identity $v - e = b_0 - b_1$ where $v$ is the number of vertices, $e$ the number of edges, and $b_0, b_1$ the (unreduced) Betti numbers of $G$ [15, Theorem 3.4]. Since $K_n$ has a total of $\frac{n(n-1)}{2}$ edges, $K_n$ needs at least $\lceil \frac{n}{2} \rceil$ collapsible complexes in a cover. Thus $\mathrm{dgcat}(K_n) \geq \lceil \frac{n}{2} \rceil - 1$ (this turns out to be the exact value of $\mathrm{dgcat}(K_n)$. See [11]). It is known that for $X$ a path-connected, finite CW complex, $\mathrm{gcat}(X)$ is bounded above by $\dim(X)$ [3, Proposition 3.2.3]. Hence $\mathrm{gcat}(|K_n|) \leq 1$. Thus the difference $\mathrm{dgcat}(K_n) - \mathrm{gcat}(|K_n|)$ can be made arbitrarily large.

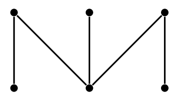
Restrictions on the graph $G$ bound the discrete category. The following result, restated in our language, is due to Nash-Williams.

**Theorem 22.** [17] *Let $G$ be a planar graph. Then* $\mathrm{dgcat}(G) \leq 2$.

*Example* 23. It is not true, however, that a nonplanar graph has discrete category necessarily greater than 1. To see this, let $G = K_{3,3}$. The following two sets are easily seen to be collapsible and to cover $K_{3,3}$:



and



Thus $\text{dgcat}(G) = 1$ but $G$ is nonplanar.

3.5. **Discrete Lusternik–Schnirelmann Theorem.** The goal of this section is to prove a discrete version of the Lusternik–Schnirelmann theorem. We begin by defining a notion of equivalence of discrete Morse functions and arguing that any discrete Morse function is equivalent to a discrete Morse function whose critical values are all distinct. Such a discrete Morse function is called **excellent**. Our reference for the definitions and basics of discrete Morse theory is [7].

**Definition 24.** [8] Two discrete Morse functions $f$ and $g$ on $K$ are said to be **equivalent** if for every pair of simplices $\sigma \subseteq \tau$ in $K$ satisfying $\dim(\sigma) = p$ and $\dim(\tau) = p + 1$, we have $f(\sigma) < f(\tau)$ if and only if $g(\sigma) < g(\tau)$.

**Lemma 25.** *Let $f\colon K \to \mathbb{R}$ be a discrete Morse function. Then there is an excellent discrete Morse function $g\colon K \to \mathbb{R}$ with all the same critical simplices of $f$ which is equivalent to $f$.*

*Proof.* Let $\sigma_1, \sigma_2 \in K$ be critical simplices such that $f(\sigma_1) = f(\sigma_2)$. If no such simplices exist, then we are done. Otherwise, define $f'\colon K \to \mathbb{R}$ by $f'(\tau) = f(\tau)$ for all $\tau \neq \sigma_1$ and $f'(\sigma_1) = f(\sigma_1) + \epsilon$ where $f(\sigma_1) + \epsilon$ is strictly less than the smallest value of $f$ greater than $f(\sigma_1)$. Then $\sigma_1$ is a critical simplex of $f'$ and $f'$ is equivalent to $f$. Repeat the construction for any two simplices of $f'$ that share the same critical value. Since $f$ has a finite number of critical values, the process terminates with an excellent discrete Morse function $g$ which is equivalent to $f$. □

The proof of the following Theorem is inspired by [3, Section 1.3]. Recall that for all $a \in \mathbb{R}$ if $L$ is a simplicial complex with discrete Morse function $f$, then the $a^{th}$ **level subcomplex** $L(a)$ is the subcomplex of $L$ consisting of all simplices $\tau$ with $f(\tau) \leq a$ as well as their faces. In other words,

$$L(a) = \bigcup_{f(\tau)\leq c} \bigcup_{\sigma \subseteq \tau} \sigma.$$

**Theorem 26.** *Let $f\colon K \to \mathbb{R}$ be a discrete Morse function with $m$ critical values. Then* $\mathrm{dgcat}(K) + 1 \leq m$.

*Proof.* By Lemma 25, we may assume that any discrete Morse function on $K$ is excellent since we are only concerned with the number of critical values. We first claim that $c_n := \min\{a \in \mathbb{R} : \exists L(a) \subseteq K \text{ s.t. } \mathrm{dgcat}_K(L(a)) \geq n-1\}$ is a critical value of $f$. Suppose by contradiction that $c_n$ is a regular value of $f$. Then by [5, Theorem 3.3] there is an $\epsilon > 0$ such that $K(c_n + \epsilon) \searrow K(c_n - \epsilon)$ in $K$. Hence $\mathrm{dgcat}_K(K(c_n + \epsilon)) = \mathrm{dgcat}_K(K(c_n - \epsilon)) \geq n - 1$ so that $c_n > c_n - \epsilon \in \{a \in \mathbb{R} : \exists L(a) \subseteq K \text{ s.t } \mathrm{dgcat}_K(L(a)) \geq n-1\}$ contradicting the fact that $c_n$ is minimum. Thus each $c_n$ is a critical value of $f$.

We now prove by induction on $n$ that $K(c_n)$ must contain at least $n$ critical simplices. By the well-ordering principle, the set $\mathrm{Im}(f)$ has a minimum, say $f(v) = 0$ for some 0-simplex $v \in K$. For $n = 1$, $c_1 = 0$ so that $K(c_1)$ contains 1 critical simplex. For the inductive hypothesis, suppose that $K(c_n)$ contains at least $n$ critical simplices. Since $f$ is excellent, $c_n < c_{n+1}$ so that there is at least 1 new critical simplex in $f^{-1}(c_{n+1})$. Thus $K(c_{n+1})$ contains at least $n+1$ critical values. Hence if $c_1 < c_2 < \ldots < c_{\mathrm{dgcat}(K)+1}$ are the critical simplices, then $K(c_{\mathrm{dgcat}(K)+1}) \subseteq K$ contains at least $\mathrm{dgcat}(K) + 1$ critical simplices. Thus $\mathrm{dgcat}(K) + 1 \leq m$. ☐

*Example* 27. We give an example to show that is best possible in the sense that equality is attained. Indeed, let $K = S^n$. Then there is a discrete Morse function on $K$ with 2 critical values [5, Cor. 4.4 ii)]. By Example 18, $\mathrm{dgcat}(S^n) = 1$ and hence $\mathrm{dgcat}(S^n) + 1 = 2 \leq 2$, showing that the inequality obtained from Theorem 26 is best possible.

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