Final: **Math 214 - Differential Equations - Fall 2002** Dr. Yahdi

To receive credit for your answers you must show all your work, explain your reasoning carefully and clearly, and include all steps necessary to completely justify each answer. Any variables you use must be clearly identified. **Box** ***in your answers when it is possible. Each problem worths 10 points. Good luck!***

Problem 1. Find the solution (explicit if possible) and its domain of definition for each initial value problem.

a) $\dfrac{dy}{dt} = -\dfrac{y}{t+5} + t$ with $y(0) = 1$

b) $\dfrac{dy}{dt} = \dfrac{-t}{y-2}$ with $y(0) = -2$

Problem 2. Suppose the following differential equation is a model for a chemical reaction:

$$\frac{dy}{dt} = 2y^4 - 3y^3 - 5y^2$$

a) Sketch the phase line for this equation, and classify each equilibrium point as a node, source, or sink.

was in test 1

b) In the same system of axes, sketch the graphs of the particular solutions y_1, y_2, y_3, y_4 of this differential equation satisfying the initial conditions (don't forget to label the axes!):

$$y_1(0) = 1, \quad y_2(0) = 3, \quad y_3(-2) = -0.5, \quad y_4(1) = -2$$

c) Describe the long-term behavior of each of the particular solutions above (i.e. $\lim_{t\to+\infty} y(t)$).

Problem 3. Consider the first order autonomous differential equation

$$\frac{dy}{dt} = 2y^4 - 3y^3 - 5y^2$$

(a) Sketch the phase line of the differential equation; use it to find and classify all the equilibrium points of the differential equation.

(b) Let $y(t)$ be the solution to the above equation such that $y(-2) = -0.43$. Find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$.

Problem 4. Consider the family of differential equations: $\dfrac{dy}{dt} = y^2 - ay + 4$ with parameter a. Locate the bifurcation values for a explicitly. Draw the bifurcation diagram with several representations of the phase lines for values of the parameter a near the bifurcation values. (Each phase line should have direction arrows.)

Problem 5. The biological relationship between two species x and y can be described as:

A= x the predator and y the prey.
B= y the predator and x the prey.
C= Species x and y cooperate.
D= Species x and y are in competition.
E= No interaction.

Select from the list above the one which best describes the relationship between species x an y in each of the following three systems:

$$\begin{cases} \dfrac{dx}{dt} = 6x - x^2 + 3xy \\ \dfrac{dy}{dt} = y + 2xy - 2y^2 \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = (-2 - x + y)x \\ \dfrac{dy}{dt} = (4 - x + 0.5y)y \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - xy \\ \dfrac{dy}{dt} = 1 - y^2 - xy \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = (2 - x + 4y)x \\ \dfrac{dy}{dt} = (1 + 3x - y)y \end{cases}$$

Problem 6. [15*pts.*] Find the particular solution of dumped harmonic oscillator: $9\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + y = 0$, with the initial conditions: $y(0) = 1, y'(0) = 1$.

$$2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 3y = 0$$

Problem 7. Find all the bifurcation values of the parameter α in the following family of differential systems, and sketch the curve represented by the system in the trace-determinant plane.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2\alpha & -1 \\ \alpha^2 + 4 & -\alpha \end{pmatrix} \mathbf{Y} \quad -\infty < a < \infty$$

Problem 8. Consider the system

$$\frac{dx}{dt} = 3 - 2x - y$$

$$\frac{dy}{dt} = x^2 - y$$

(a) Give the linearized system near each equilibrium point.

(b) Classify each equilibrium point of the system and sketch its representative phase plane.

Problem 9. Give an expression for the following function using the Heaviside function $u_a(t)$.

$$f(t) = \begin{cases} 5 & \text{if} t < \pi \\ \sin t & \text{if} \pi \le t < 8 \\ t^2 & \text{if} t \ge 8 \end{cases}$$

$$f(t) = \begin{cases} 5 & \text{if} t < 2\pi \\ \cos(3t) & \text{if} 2\pi \le t < 7 \\ t^2 - 4t & \text{if} t \ge 7 \end{cases}$$

Problem 10. Find the following Laplace transforms.

(a) $\mathcal{L}\{2(t-2)^2 + 3\sin(3t)\}$

(b) $\mathcal{L}\{2(t-2)^2 u_2(t) + 3e^{4t}\cos(3t)\}$

(a) $\mathcal{L}\left\{-6e^{3(t-4)}\sin 4(t-4)u_4(t)\right\}$

(a) $\mathcal{L}\left\{(2t-1)u3(t)+5e^{-3t}\cos(4t)\right\}$

(a) $\mathcal{L}\left\{-6e^{3(t-4)}\sin 4(t-4)u_4(t)\right\}$

a) $\mathcal{L}\left\{2(t-2)^2u_2(t)\right\}$

Good a) $\mathcal{L}\left\{t^2u_1(t)\right\}$

Good b) $\mathcal{L}\left\{e^t\sin(t-2)u_2(t)\right\}$

Problem 11. Find the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-1)(s-2)}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-4s+8}\right\}$

b) $\mathcal{L}^{-1}\left\{\frac{e^{-3s}(3s+2)}{s^2-4s+8}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{1-s}{s^2-6s+13}\right\}$

Good a) $\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+8}\right\}$

Good b $\mathcal{L}^{-1}\left\{\frac{e^{-3s}(2s+1)}{s^2-4s+8}\right\}$

Problem 12. Use Laplace transforms to solve each differential equation.

(a) $\frac{dy}{dt}+9y=u_5(t),\ y(0)=-2$

(b) $\frac{d^2y}{dt^2}+4y=3\cos(t),\ y(0)=0,\ y'(0)=0$

(c) $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+2y=e^{-t},\ y(0)=0,\ y'(0)=1$

(d) $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+(w^2+1)y=h(t),\ y(0)=0,\ y'(0)=0$

(a) $\frac{dy}{dt}=-9y+u_5(t),\ y(0)=-2$

(b) $\frac{d^2y}{dt^2}+4y=3\sin t,\ y(0)=1,\ y'(0)=0$