

Sections 10.3 & 10.4 : Optimization problems

How to solve an optimization problem?

1. **Step 1:** Understand the problem and underline what is important (what is known, what is unknown, what we are looking for, *dots*)
2. **Step 2:** Draw a “diagram”; if it is possible.
3. **Step 3:** Assign “symbols” or “variables” for all the quantities involved (know or unknown), and label the diagram.
4. **Step 4:** Write the quantity **Q** to be maximized or minimized in terms of some of the previous variables (from Step 3). Example: $Q = g(x, y, h)$
5. **Step 5:** Rewrite **Q** as a function of **only one variable**. To do this, find the relationships between the variables using the information given in the problem. Then, use these equations to eliminate all but one of the variables in the expression of Q . Thus, we get $Q = f(x)$.
6. **Step 5:** Use the methods of sections 10.1 and 10.2 to find the maximum or the minimum of the quantity $Q = f(x)$.
7. **REMARK:** Do not forget to find the endpoints and to check if the maximum or the minimum is at these points if you have more than one critical number in the domain.
8. **Short-cut:** If there is **only one critical** number **a** in the domain, then :

$$\begin{cases} \text{if we have a local maximum at } \mathbf{a} & \implies \text{we have a } \mathbf{global} \text{ maximum at } \mathbf{a}, \\ \text{if we have a local minimum at } \mathbf{a} & \implies \text{we have a } \mathbf{global} \text{ minimum at } \mathbf{a}. \end{cases}$$

In other term, we do not need to check the position of the endpoints.

9. **Reminder:** At the worksheet I gave you in the beginning of the semester (it is the KEY FORMULAS for Chapter 9 posted at the homework assignment web page) of the textbook, you can find all the formulas related to the cost, revenue and profit. You should work

Examples:

Problem 1. The regular air fare between Boston and San Francisco is \$500. An airline using planes with a capacity of 300passengers on this route observes that they fly with an average of 180 passengers. Market research tells the airlines’ managers that each \$ 5 fare reduction would attract, on average, 3 more passengers for each flight. How should they set the fare to maximize their revenue? Explain your reasoning to receive credit.

- **SOLUTION:**

Let R = the revenue function = quantity \times price

Let n = the number of times the fare is reduced by \$ 5 dollars. Then:

$$\begin{cases} \text{price} & = \$500 - n \cdot (\$5) = 500 - 5n \text{ dollars,} \\ \text{quantity} & = \text{number of passengers} = 180\text{passengers} + n \cdot (3\text{passengers}) = 180 + 3n \text{ passengers.} \end{cases}$$

Hence $R(n) = (180 + 3n)(500 - 5n) = 90\,000 + 600n - 15n^2$ to maximize (for $0 \leq n \leq 40$).

$R'(n) = 600 - 30n = 0 \Rightarrow n = \frac{600}{30} = 20$ is the only critical number.

$R''(n) = -30 \Rightarrow R''(20) = -30 < 0$. By the second derivative test, R has a local maximum at $n = 20$, which is an absolute maximum since it is the only critical number.

The best fare to maximize the revenue is then: $\$ 500 - 5(20) = \400 , with $180 + 3(20) = 240$ passengers and a revenue of $R(20) = \$96,000$.

Problem 2. A baseball team plays in a stadium that hold 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. A market survey showed that for each \$0.10 decrease in the ticket prices, on the average, the attendance will increase by 300. How should ticket prices be set to maximize revenue?

• SOLUTION:

Let R = the revenue function = quantity \times price

Let n = the number of times the price of the ticket is reduced by \$ 0.10. Then:

$$\begin{cases} \text{price} & = \$10 - n \cdot (\$0.10) = 10 - 0.10n \text{ dollars,} \\ \text{quantity} & = \text{number of spectators} = 27,000 \text{ spectators} + n \cdot (300 \text{ spectators}) = 27,000 + 300n \text{ spectators.} \end{cases}$$

Hence $R(n) = (27,000 + 300n)(10 - 0.10n) = 270,000 + 300n - 30n^2$ to maximize.

$R'(n) = 300 - 60n = 0 \Rightarrow n = \frac{300}{60} = 5$ is the only critical number.

$R''(n) = -60 \Rightarrow R''(5) = -60 < 0$. By the second derivative test, R has a local maximum at $n = 5$, which is an absolute maximum since it is the only critical number.

The best ticket prices to maximize the revenue is then: $\$ 10 - 0.10(5) = 9.50 \$$,

with $27,000 + 300(5) = 28,500$ spectators and a revenue of $\$ R(5) = 270,750$.

Problem 3. A Florida Citrus grower estimates that if 60 orange trees are planted; the average yield per tree will be 400 oranges. The average yield will decrease by 4 oranges per tree for each additional tree planted on the same acreage. How many trees should the grower plant to maximize the total yield?

• SOLUTION:

Let n = the number of additional trees. Let Y = the total yield = number of trees \times the yield per tree. Then:

$$Y(n) = (60 \text{ trees} + n \cdot \text{trees})(400 \text{ oranges} - n \cdot 4 \text{ oranges}) = (60 + n)(400 - 4n) = 24,000 + 160n - 4n^2$$

to maximize ! Lets find the critical numbers:

$Y'(n) = 160 - 8n = 0 \Rightarrow n = \frac{160}{8} = 20$ is the only critical number.

Moreover, $Y''(n) = -8 \Rightarrow Y''(20) = -8 < 0$. By the second derivative test, Y has a local maximum at $n = 20$, which is an absolute maximum since it is the only critical number.

The grower should plant $60 + 20 = 80$ trees to maximize the total yield.

Problem 4. A manufacturer of men's shirts determines that her costs will be 500 dollars for overhead plus 9 dollars for each shirt made. Her accountant has estimated that her selling price p should be determined by

$$p = 30 - 0.2\sqrt{x} \quad \text{where } x \text{ is the number of shirts sold.}$$

1. Give the formula for the profit function.

SOLUTION: Profit = $\mathbb{P}(x) = R(x) - C(x)$, where

$$\begin{cases} R(x) = & \text{Revenue} = \text{Quantity} \times \text{Price} \\ C(x) = & \text{Cost function} \end{cases}$$

Since the Quantity = is the number of shirts sold = x and the price $p = 30 - 0.2\sqrt{x}$, then

$$\begin{cases} R(x) & = x \cdot (30 - 0.2\sqrt{x}) = 30x - 0.2x^{\frac{3}{2}} \\ C(x) & = \$500 + \$9 \cdot \text{number of shirts} = 500 + 9x \end{cases}$$

Hence $\mathbb{P}(x) = 30x - 0.2x^{\frac{3}{2}} - (500 + 9x) = 21x - 0.2x^{\frac{3}{2}} - 500$.

2. **How many shirts should be produced to maximize profit?**

SOLUTION: To maximize the $\mathbb{P}(x)$, we need to find the critical numbers:

$\mathbb{P}'(x) = 21 - 0.3x^{\frac{1}{2}} = 0 \Rightarrow \sqrt{x} = \frac{21}{0.3} = 70$. Therefore, $x = 4900$ is the only critical number.

Moreover, $\mathbb{P}''(x) = -0.15x^{-\frac{1}{2}} \Rightarrow \mathbb{P}''(4900) = -0.15 \cdot 4900^{-\frac{1}{2}} = -\frac{0.15}{70} < 0$. By the second derivative test, R has a local maximum at $x = 4900$, which is an absolute maximum since it is the only critical number.

3. **At what price will the shirts be sold?**

SOLUTION: The best price to maximize the profit is then: $p(4900) = 30 - 0.2\sqrt{4900} = \16 .

4. **What is her resulting profit?**

SOLUTION: The corresponding profit is $\mathbb{P}(4900) = \$33800$

Problem 5. Farmers can get 2 dollars per bushel for their potatoes on July 1, and after that, the price drops by 2 cents per bushel per extra day. On July 1, a farmer had 80 bushels of potatoes in the field and estimates that the crop is increasing at the rate of 1 bushel per day. When should the farmer harvest the potatoes to maximize his revenue?

• SOLUTION:

Let x = the number of extra days after July 1.

Let R = the revenue = quantity \times price. Then:

$$\begin{cases} \text{price} & = \$2 - x \cdot (\$0.02) = 2 - 0.02x \text{ dollars,} \\ \text{quantity} & = \text{number of bushels} = 80 + x \cdot 1 \text{ bushel per day} = 80 + x \text{ bushels.} \end{cases}$$

Hence $R(x) = (80 + x)(2 - 0.02x) = 160 + 0.4x - 0.02x^2$ to maximize ! Lets find the critical numbers:

$R'(x) = 0.4 - 0.04x = 0 \Rightarrow x = \frac{0.4}{0.04} = 10$ is the only critical number.

Moreover, $R''(x) = -0.04 \Rightarrow R''(10) = -0.04 < 0$. By the second derivative test, R has a local maximum at $x = 10$, which is an absolute maximum since it is the only critical number.

The farmer should harvest the potatoes 10 extra days after July 1, so on **July 11**.

Problem 6. A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden, She will use shrubs costing \$25 per foot along three sides and fencing costing \$10 per foot along the fourth side, Find the minimum total cost.

• SOLUTION: If the rectangular region has dimensions x and y , then its area is $A = xy = 3000 \text{ft}^2$. So

$$y = \frac{3000}{x}$$

If y is the side with fencing costing \$10 per foot, then the cost for this side is **\$ 10 y**.

The cost for the three other sides, where shrubs costing \$15 is used, is then **\$ 15 (2x+y)**.

Therefore the total cost is: $C(x) = 10y + 15(2x + y) = 30x + 25y$.

Since $y = \frac{3000}{x}$, then $C(x) = 30x + 25\frac{3000}{x}$ that we wish to minimize.

Since $C'(x) = 30 - 25\frac{3000}{x^2}$, then $C'(x) = 0$ for $x^2 = 25\frac{3000}{30} = 2500$. Therefore, since x is positive, we have only one critical number in the domain which is $x = 50 \text{ft}$.

Since $C''(x) = 25\frac{1500}{x^3}$, we have $C''(50) > 0$. Thus, by the 2nd derivative test, C has a local minimum at $x = 50$, and therefore an absolute minimum because we have only one critical number in the domain. Hence, the minimum cost is $C(50) = \$4500$, with the dimensions $x = 50 \text{ft}$ and $y = \frac{3000}{50} = 60 \text{ft}$.

More Problems

Problem 7. It is estimated that the cost of constructing an office building that is n floors high is $c(n) = 2n^2 + 500n + 600$ thousand dollars. How many floors should the building have in order to minimize the **average cost** per floor? (*Answer: 17 floors*)

Problem 8. Find the quantity q of items which maximizes the profit if it is not possible to produce more than 800 items, and if the total revenue and the total cost (in dollars) are given below: (*Answer: \$650*)

$$R(q) = 5q - 0.003q^2 \quad \text{and} \quad C(q) = 300 + 1.1q$$

Problem 9. An apartment complex has 400 apartments. At \$ 400 per month for each apartment all the apartment will be occupied. Each \$20 increase will produce 10 vacancies. Let x denote the number of \$ 20 increase. What is the largest revenue that the apartment complex can generate? (*Answer: $x = 10$*)

Problem 10. Suppose you run a small independent furniture business. You sign a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be \$80 per chair up to 300 chairs, and above 300, for every additional chair over 300 ordered, the price will be reduced by \$0.20 per chair **on the whole order**. What are the largest revenue your company can make if the customer choose to order more than 300 chairs?

Problem 11. There are 320 yards of fencing available to enclose a rectangular field. How should this fencing be used so that the enclosed area is as large as possible? (80×80)

Problem 12. A city recreation department plans to build a rectangular playground having an area of 3600 square meters and surround it by a fence. How can this be done using the least amount of fencing? (*Answer: 60×60 meters*)

Problem 13. You are asked to design a cylindrical aluminum can holding a volume of 300cm^3 . If both the top and the bottom of the can are **twice as thick** as the sides of the can, what dimensions of the can will minimize the amount needed. (A cylinder with radius r and height h has a volume given by $V = \pi r^2 h$, its surface area (lids not included) is $S = 2\pi r h$.)